

AN INTEGRAL FUNCTIONAL DRIVEN BY FRACTIONAL BROWNIAN MOTION*

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ABSTRACT. Let B^H be a fractional Brownian motion with Hurst index $0 < H < 1$ and the weighted local time $\mathcal{L}^H(\cdot, t)$. In this paper, we consider the integral functional

$$\mathcal{C}_t^H(a) := \lim_{\varepsilon \downarrow 0} \int_0^t 1_{\{|B_s^H - a| > \varepsilon\}} \frac{1}{B_s^H - a} ds^{2H} \equiv \frac{1}{\pi} \mathcal{H} \mathcal{L}^H(\cdot, t)(a)$$

in $L^2(\Omega)$ with $a \in \mathbb{R}, t \geq 0$ and \mathcal{H} denoting the Hilbert transform. We show that

$$\mathcal{C}_t^H(a) = 2 \left((B_t^H - a) \log |B_t^H - a| - B_t^H + a \log |a| - \int_0^t \log |B_s^H - a| \delta B_s^H \right)$$

for all $a \in \mathbb{R}, t \geq 0$ which is the fractional version of Yamada's formula, where the integral is the Skorohod integral. Moreover, we introduce the following *occupation type formula*:

$$\int_{\mathbb{R}} \mathcal{C}_t^H(a) g(a) da = 2H\pi \int_0^t (\mathcal{H}g)(B_s^H) s^{2H-1} ds$$

for all continuous functions g with compact support.

1. INTRODUCTION

Given $H \in (0, 1)$, a fractional Brownian motion (fBm) $B^H = \{B_t^H, 0 \leq t \leq T\}$ with Hurst index H is a mean zero Gaussian process such that

$$E[B_t^H B_s^H] = \frac{1}{2} [t^{2H} + s^{2H} - |t - s|^{2H}]$$

for all $t, s \geq 0$. For $H = 1/2$, B^H coincides with the standard Brownian motion B . B^H is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with B^H . As a Gaussian process, one can construct the stochastic calculus of variations with respect to B^H . Some surveys and complete literatures for fBm could be found in Biagini *et al* [5], Decreusefond-Üstünel [12], Hu [19], Mishura [24], Nourdin [25], Nualart [26] and the references therein.

Let now F be an absolutely continuous function such that the Skorohod integral

$$\int_0^t F'(B_s^H - a) \delta B_s^H$$

is well-defined and the second derivative $F'' = f$ exists in the sense of Schwartz's distribution. Then the process

$$(1.1) \quad \mathcal{K}_t^H(a) := 2 \left(F(B_t^H - a) - F(-a) - \int_0^t F'(B_s^H - a) \delta B_s^H \right),$$

*The Project-sponsored by NSFC (11571071, 11426036) and Innovation Program of Shanghai Municipal Education Commission(12ZZ063).

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2000 *Mathematics Subject Classification*. Primary 60G15, 60H05; Secondary 60H07.

Key words and phrases. fractional Brownian motion, Malliavin calculus, local time, fractional Itô formula and Cauchy's principal value.

exists for all $a \in \mathbb{R}$. Denote

$$(1.2) \quad \mathcal{X}_t^H(a) := \int_0^t f(B_s^H - a) ds^{2H}$$

for all $t \geq 0, a \in \mathbb{R}$. By Itô's formula one can find the following questions:

- if the Lebesgue integral (1.2) converges,

$$\mathcal{K}_t^H(a) = \mathcal{X}_t^H(a) ?$$

- how to characterize the process $\mathcal{K}_t^H(a)$ if the Lebesgue integral (1.2) diverges?

Clearly, the first question is positive by approximating. However, the second question is not obvious even if $H = \frac{1}{2}$ and f is a special function. Thus, the question arises again:

- for which functions does the Lebesgue integral (1.2) diverge?

When $H = \frac{1}{2}$, B^H coincides with the standard Brownian motion B and by the Engelbert-Schmidt zero-one law, the Lebesgue integral (1.2) diverges if $F'' = f$ is not locally integrable, i.e.

$$\int_{-M}^M |f(x - a)| dx = \infty$$

for some $M > 0$. Thus, when

$$|f(x)| \geq C|x|^{-\alpha}$$

for $x \in \mathbb{R}$, the Lebesgue integral (1.2) diverges, where $C > 0, \alpha \geq 1$. For some special functions F , for examples,

$$(1.3) \quad F''(x) = |x|^{-\gamma} \text{sign}(x)$$

with $1 \leq \gamma < \frac{3}{2}$, one studied the characterization and properties of the process $\mathcal{K}_t^{\frac{1}{2}}(a)$. Itô-McKean [21] first considered the process $\mathcal{K}_t^{\frac{1}{2}}(a)$ and the Lebesgue integral (1.2) for F satisfying (1.3) with $\gamma = 1$. For the process $\mathcal{K}_t^{\frac{1}{2}}(a)$ and the Lebesgue integral (1.2) driven by the function F satisfying (1.3), some systematic studies are due to Biane-Yor [6], Yamada [29, 30, 31] and Yor [36], and some extensions and limit theorems are established by Bertoin [3, 4], Cherny [7], Csaki *et al* [9, 10], Csaki-Hu [20], Hu [20], Fitzsimmons-Gettoor [14, 15], Mansuy-Yor [23], Yor [37] and the references therein. However, those researches apply only to Markov process, and for non-Markov processes there has only been little investigation on the integral functional. See Eddahbi-Vives [13], Gradinaru *et al.* [17], Yan [32] and Yan-Zhang [35].

When $H \neq \frac{1}{2}$ the second and third questions above are not trivial. The main difficulty consists in the fact that the stochastic integral

$$(1.4) \quad \int_0^t F'(B_s^H - a) \delta B_s^H$$

is a Skorohod integral with respect to the fBm and the integrand is not smooth. Therefore, its control is not obvious and one needs sharp estimates. The L_2 norm of this stochastic Skorohod integral involves the Malliavin derivatives of the integrand and tedious estimation on the joint density of the fBm. Moreover, for a nonsmooth function f it is not easy to give an exact calculus of the moment of order 2 for the Skorohod integral (1.4) even if the simple functions $F'(x) = \log|x|$ and $F'(x) = |x|^{-\alpha} \text{sign}(x)$ with $\alpha > 0$. But, when $H = \frac{1}{2}$, the integral (1.4) is Itô's integral and its existence is obvious. On the other hand, it is unclear whether the Engelbert-Schmidt zero-one law actually holds for fBm B^H . Thus, it seems interesting to study the process $\mathcal{K}_t^H(a)$ and the Skorohod integral (1.4) with the singular integrand for $H \neq \frac{1}{2}$. In this paper, as a start reviewing the object and continued to Yan [32], we consider the integrals

$$(1.5) \quad \int_0^t \frac{ds^{2H}}{B_s^H - a}, \quad a \in \mathbb{R}$$

and the processes

$$(1.6) \quad \mathcal{C}_t^H(a) := 2 \left(F(B_t^H - a) - F(-a) - \int_0^t F'(B_s^H - a) \delta B_s^H \right), \quad a \in \mathbb{R}$$

with $t \geq 0$, where the integral in (1.6) is the Skorohod integral and $F(x) = x \log |x| - x$. In the present paper we will consider the functional and discuss some related questions. We will divide the discussion as two parts since the research method of the case $\frac{1}{2} < H < 1$ is essentially different with the case $0 < H < \frac{1}{2}$. In Section 6 we study the case $0 < H < \frac{1}{2}$ and the case $\frac{1}{2} < H < 1$ is considered in Section 3, Section 4 and Section 5.

This paper is organized as follows. In Section 2 we present some preliminaries for fBm. In Section 3, we consider the existence of $\mathcal{C}^H(a)$ for $\frac{1}{2} < H < 1$. In fact, by smoothness approximating one can prove the existence of the Skorohod integral

$$\int_0^t \log |B_s^H - a| \delta B_s^H,$$

however, it is not easy to give the exact estimates of the moments. To give the existence and exact estimates of the moments, we define the function

$$\begin{aligned} \Psi_{s,r,a,b}(x,y) &:= \varphi_{s,r}(x,y) - \varphi_{s,r}(x,b)\theta(1+b-y) \\ &\quad - \varphi_{s,r}(a,y)\theta(1+a-x) + \varphi_{s,r}(a,b)\theta(1+a-x)\theta(1+b-y) \end{aligned}$$

with $x, y, a, b \in \mathbb{R}, s, r > 0$, where $\theta(x) = 1_{\{x>0\}}$ and $\varphi_{s,r}(x,y)$ is the density function of (B_s^H, B_r^H) , and show that the identity

$$(1.7) \quad E [G'_1(B_s^H - a)G'_2(B_r^H - b)] = \int_{\mathbb{R}^2} G'_1(x-a)G'_2(y-b)\Psi_{s,r,a,b}(x,y)dx dy$$

holds for all $G_1, G_2 \in C^\infty(\mathbb{R})$ with compact supports and $G_1(1) = G_2(1) = 0$. By using (1.7) we show that the integral $\int_0^t F'_+(B_s^H - a) \delta B_s^H$ exists and

$$\begin{aligned} E \left| \int_0^t F'_+(B_s^H - a) \delta B_s^H \right|^2 &= \int_0^t \int_0^t E [F'_+(B_s^H - a)F'_+(B_r^H - a)] \phi(s,r) ds dr \\ &\quad + \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \phi(s,\eta)\phi(r,\xi) \int_a^\infty \int_a^\infty \frac{\Psi_{s,r,a,a}(x,y)}{(x-a)(y-a)} dx dy \end{aligned}$$

with $\phi(s,r) = H(2H-1)|s-r|^{2H-2}$, where the integral $\int_0^t F'_+(B_s^H) \delta B_s^H$ is the Skorohod integral and

$$F_+(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x \log x - x, & \text{if } x > 0. \end{cases}$$

In Section 4, for $\frac{1}{2} < H < 1$ we show that the representation

$$(1.8) \quad \mathcal{C}_t^H(a) = \lim_{\varepsilon \downarrow 0} \int_0^t 1_{\{|B_s^H - a| \geq \varepsilon\}} \frac{2Hs^{2H-1}}{B_s^H - a} ds, \quad a \in \mathbb{R}, t \geq 0$$

holds in $L^2(\Omega)$, which points out that $a \mapsto \frac{1}{\pi} \mathcal{C}_t^H(a)$ coincides with the Hilbert transform of the weighted local time

$$a \mapsto \mathcal{L}^H(a, t) = 2H \int_0^t \delta(B_s^H - a) s^{2H-1} ds$$

and the fractional version of Yamada's formula

$$(B_t^H - a) \log |B_t^H - a| - (B_t^H - a) = -a \log |a| + a + \int_0^t \log |B_s^H - a| \delta B_s^H + \frac{1}{2} \mathcal{C}_t^H(a)$$

holds. In section 5 we introduce the so-called *occupation type formula*

$$(1.9) \quad \int_{\mathbb{R}} \mathcal{C}_t^H(a) g(a) da = 2H\pi \int_0^t (\mathcal{H}g)(B_s^H) s^{2H-1} ds$$

for all continuous function g with compact support and $\frac{1}{2} < H < 1$, where \mathcal{H} denotes Hilbert transform. In Section 6 we study the case $0 < H < \frac{1}{2}$ by using the *generalized quadratic covariation* introduced in Yan *et al* [33].

2. PRELIMINARIES

2.1. Cauchy principal value. It is known that the Cauchy principal value, named after Augustin Louis Cauchy, is a method for assigning values to certain improper integrals which would otherwise be undefined. Depending on the type of singularity in the integrand f , the Cauchy principal value is defined as one of the following:

$$(2.1) \quad \lim_{\varepsilon \downarrow 0} \left(\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx \right) = \lim_{\varepsilon \downarrow 0} \int_a^b 1_{\{|c-x| \geq \varepsilon\}} f(x),$$

where $c \in (a, b)$ is a unique point such that

$$\int_a^b f(x)dx = \infty.$$

The limiting operation given in (2.1) is called the (Cauchy) principal value of the divergent integral $\int_a^b f(x)dx$ and the limiting process displayed in (2.1) is denoted as

$$\text{v.p.} \int f(x)dx.$$

The notation v.p. (valeur principale) is seen in European writings. We have, as an example,

$$\text{v.p.} \int_a^b \frac{dx}{c-x} = \log \frac{c-a}{b-c}$$

for all $a < c < b$. Moreover, for a Borel function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ with

$$\int_a^{a+1} \frac{|\varphi(x) - \varphi(a)|}{x-a} dx + \int_{a+1}^\infty \frac{|\varphi(x)|}{x-a} dx < \infty,$$

we can define the Cauchy's principal value

$$\begin{aligned} \text{v.p.} \int_a^\infty \frac{\varphi(x)}{x-a} dx &:= \int_a^{a+1} \frac{\varphi(x) - \varphi(a)}{x-a} dx + \int_{a+1}^\infty \frac{\varphi(x)}{x-a} dx \\ &= \lim_{\varepsilon \downarrow 0} \left(\int_{a+\varepsilon}^\infty \frac{\varphi(x)}{x-a} dx + \varphi(a) \log \varepsilon \right). \end{aligned}$$

Recall that the Hilbert transform $\mathcal{H}f$ of $f \in L^2(\mathbb{R})$ is defined as follows

$$(2.2) \quad \mathcal{H}f(a) := \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} 1_{\{|x-a| \geq \varepsilon\}} \frac{f(x)dx}{x-a} = \frac{1}{\pi} \text{v.p.} \int_{\mathbb{R}} \frac{f(x)dx}{x-a} \equiv \frac{1}{\pi} \text{v.p.} \frac{1}{x} * f(x),$$

where $*$ denotes the convolution in the theory of distributions, which plays an important role in real and complex analysis. It is also important to note that $\mathcal{H}f$ belongs to L^2 and

$$\int_{\mathbb{R}} (\mathcal{H}f(x))^2 dx = \int_{\mathbb{R}} f^2(x) dx$$

holds, and moreover, if f is a Hölder continuous function with compact support, then the limit in (2.2) exists for every $x \in \mathbb{R}$. For more aspects on these material we refer to King [22].

2.2. Fractional Brownian motion. In this subsection, we briefly recall some basic definitions and results of fBm. For more aspects on these material we refer to Alós et al. [1], Biagini *et al* [5], Decreusefond-Üstünel [12], Hu [19], Mishura [24], Nourdin [25], Nualart [26] and the references therein. Throughout this paper we assume that $0 < H < 1$ is arbitrary but fixed and we let $B^H = \{B_t^H, 0 \leq t \leq T\}$ be a one-dimensional fBm with Hurst index H defined on $(\Omega, \mathcal{F}^H, P)$.

Let \mathcal{H} be the completion of the linear space \mathcal{E} generated by the indicator functions $1_{[0,t]}, t \in [0, T]$ with respect to the inner product

$$\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathcal{H}} = \frac{1}{2} [t^{2H} + s^{2H} - |t - s|^{2H}].$$

The application $\varphi \in \mathcal{E} \rightarrow B^H(\varphi)$ is an isometry from \mathcal{E} to the Gaussian space generated by B^H and it can be extended to \mathcal{H} . When $\frac{1}{2} < H < 1$ the Hilbert space \mathcal{H} can be written as

$$\mathcal{H} = \{\varphi : [0, T] \rightarrow \mathbb{R} \mid \|\varphi\|_{\mathcal{H}} < \infty\},$$

where

$$\|\varphi\|_{\mathcal{H}}^2 := \int_0^T \int_0^T \varphi(s)\varphi(r)\phi(s, r)dsdr$$

with $\phi(s, r) = H(2H - 1)|s - r|^{2H-2}$. Notice that the elements of the Hilbert space \mathcal{H} may not be functions but distributions of negative order (see, for instance, Pipiras-Taqqu [27]). Denote by \mathcal{S} the set of smooth functionals of the form

$$(2.3) \quad F = f(B^H(\varphi_1), B^H(\varphi_2), \dots, B^H(\varphi_n)),$$

where $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its derivatives are bounded) and $\varphi_i \in \mathcal{H}$. The *derivative operator* D^H (the Malliavin derivative) of a functional F of the form (2.3) is defined as

$$D^H F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(B^H(\varphi_1), B^H(\varphi_2), \dots, B^H(\varphi_n))\varphi_j.$$

The derivative operator D^H is then a closable operator from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$. We denote by $\mathbb{D}^{1,2}$ the closure of \mathcal{S} with respect to the norm

$$\|F\|_{1,2} := \sqrt{E|F|^2 + E\|D^H F\|_{\mathcal{H}}^2}.$$

The *divergence integral* δ^H is the adjoint of derivative operator D^H . That is, we say that a random variable u in $L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator δ^H , denoted by $\text{Dom}(\delta^H)$, if

$$E|\langle D^H F, u \rangle_{\mathcal{H}}| \leq c\|F\|_{L^2(\Omega)}$$

for every $F \in \mathcal{S}$. In this case $\delta^H(u)$ is defined by the duality relationship

$$(2.4) \quad E[F\delta^H(u)] = E\langle D^H F, u \rangle_{\mathcal{H}}$$

for any $u \in \mathbb{D}^{1,2}$. We have $\mathbb{D}^{1,2} \subset \text{Dom}(\delta^H)$, and when $\frac{1}{2} < H < 1$ we have

$$(2.5) \quad E[\delta^H(u)^2] = E\|u\|_{\mathcal{H}}^2 + E \int_{[0,T]^4} D_\xi^H u_r D_\eta^H u_s \phi(\eta, r)\phi(\xi, s)dsdrd\xi d\eta$$

for any $u \in \mathbb{D}^{1,2}$. By the duality between D^H and δ^H one have that the following result for the convergence of divergence integrals which is given in Nualart [26].

Proposition 2.1. *Let $\{u_n, n = 1, 2, \dots\} \subset \text{Dom}(\delta^H)$ such that $u_n \rightarrow u$ in $L^2(\Omega; \mathbb{H})$ for some $u \in L^2(\Omega; \mathbb{H})$. If that there exists $U \in L^2(\Omega)$ such that*

$$\delta^H(u_n) \longrightarrow U$$

in $L^2(\Omega; \mathbb{H})$, as $n \rightarrow \infty$. Then, u belongs to $\text{Dom}(\delta^H)$ and $\delta^H(u) = U$.

We will use the notation

$$\delta^H(u) = \int_0^T u_s \delta B_s^H$$

to express the Skorohod integral of a process u , and the indefinite Skorohod integral is defined as $\int_0^t u_s \delta B_s^H = \delta^H(u1_{[0,t]})$. Recall the Itô type formula for fBm B^H ,

$$f(B_t^H) = f(0) + \int_0^t f'(B_s^H) \delta B_s^H + H \int_0^t f''(B_s^H) s^{2H-1} ds$$

for any $f \in C^2(\mathbb{R})$. Also recall that B^H has a local time $\mathcal{L}^H(x, t)$ continuous in $(x, t) \in \mathbb{R} \times [0, \infty)$ which satisfies the occupation formula (see Geman-Horowitz [16])

$$(2.6) \quad \int_0^t \Phi(B_s^H) ds = \int_{\mathbb{R}} \Phi(x) \mathcal{L}^H(x, t) dx$$

for every nonnegative bounded function Φ on \mathbb{R} , and such that

$$\mathcal{L}^H(x, t) = \int_0^t \delta(B_s^H - x) ds = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \lambda(s \in [0, t], |B_s^H - x| < \epsilon),$$

where λ denotes Lebesgue measure and $\delta(x)$ is the Dirac delta function. It is well-known that the local time $\mathcal{L}^H(x, t)$ has Hölder continuous paths of order $\gamma \in (0, 1 - H)$ in time, and of order $\kappa \in (0, \frac{1-H}{2H})$ in the space variable, provided $H \geq \frac{1}{3}$. Define the so-called weighted local time $\mathcal{L}^H(x, t)$ of B^H at x as follows

$$\mathcal{L}^H(x, t) = 2H \int_0^t s^{2H-1} \mathcal{L}^H(x, ds) \equiv 2H \int_0^t \delta(B_s^H - x) s^{2H-1} ds.$$

The Hölder continuity properties of $\mathcal{L}^H(x, t)$ can be transferred to the weighted local time $\mathcal{L}^H(x, t)$, and then \mathcal{L}^H has a compact support in x , and the following Tanaka formula holds (see Coutin *et al* [8] and Hu *et al* [18]):

$$(2.7) \quad (B_t^H - x)^- = (-x)^- - \int_0^t 1_{\{B_s^H < x\}} \delta B_s^H + \frac{1}{2} \mathcal{L}^H(x, t).$$

At the end of this section we will establish some technical estimates associated with fractional Brownian motion. For simplicity we let C stand for a positive constant depending only on the subscripts and its value may be different in different appearance, and this assumption is also adaptable to c .

Lemma 2.1. *For all $r, s \in [0, T]$, $s \geq r$ and $0 < H < 1$ we have*

$$(2.8) \quad \frac{1}{2} (2 - 2^H) r^{2H} (s - r)^{2H} \leq s^{2H} r^{2H} - \mu_{s,r}^2 \leq 2r^{2H} (s - r)^{2H},$$

where $\mu_{s,r} = E(B_s^H B_r^H)$.

By the local nondeterminacy of fBm we can prove the lemma (Yan *et al* [34]), and Yan *et al* [33] gave an elementary proof by using the inequality

$$(2.9) \quad (1 + x)^\alpha \leq 1 + (2^\alpha - 1)x^\alpha$$

with $0 \leq x, \alpha \leq 1$. It is important to note that inequality (2.9) is stronger than the well known (Bernoulli) inequality

$$(1 + x)^\alpha \leq 1 + \alpha x^\alpha \leq 1 + x^\alpha,$$

because $2^\alpha - 1 \leq \alpha$ for all $0 \leq \alpha \leq 1$.

Lemma 2.2. *For all $s > r > 0$ and $\frac{1}{2} < H < 1$ we have*

$$(2.10) \quad c_H (s - r) r s^{2H-2} \leq \mu - r^{2H} \leq C_H (s - r) r s^{2H-2}$$

and

$$(2.11) \quad c_H (s - r) s^{2H-1} \leq s^{2H} - \mu \leq C_H (s - r) s^{2H-1}$$

where $\mu_{s,r} = E(B_s^H B_r^H)$.

Proof. For the inequalities (2.11) we have

$$\mu - r^{2H} = \frac{1}{2} (s^{2H} - r^{2H} - (s-r)^{2H}) = \frac{1}{2} s^{2H} (1 - x^{2H} - (1-x)^{2H})$$

with $x = \frac{r}{s}$. By the continuity of the functions

$$f_1(x) = \frac{1 - x^{2H} - (1-x)^{2H}}{x(1-x)}, \quad f_2(x) = \frac{x(1-x)}{1 - x^{2H} - (1-x)^{2H}}$$

for $x \in (0, 1)$ and $\lim_{x \rightarrow 0} f_i(x) = \lim_{x \rightarrow 1} f_i(x) = 2H$ for $i = 1, 2$, we see that there exists a constant $C > 0$ such that

$$\frac{1}{C} x(1-x) \leq 1 - x^{2H} - (1-x)^{2H} \leq C x(1-x)$$

for all $x \in [0, 1]$, which gives the inequalities (2.10). The inequalities (2.11) is clear. \square

3. THE EXISTENCE OF $\mathcal{C}^H(a)$

Beside on the smooth approximation one can prove the existence of \mathcal{C}^H . In order to use the smooth approximation, we define the function $(x, y) \mapsto \Psi_{s,r,a,b}(x, y)$ on \mathbb{R}^2 by

$$\begin{aligned} \Psi_{s,r,a,b}(x, y) := & \varphi_{s,r}(x, y) - \varphi_{s,r}(x, b)\theta(1+b-y) \\ & - \varphi_{s,r}(a, y)\theta(1+a-x) + \varphi_{s,r}(a, b)\theta(1+a-x)\theta(1+b-y) \end{aligned}$$

with $s, r > 0$ and $a, b \in \mathbb{R}$, where $\theta(x) = 1_{\{x>0\}}$ and $\varphi_{s,r}(x, y)$ denotes the density function of (B_s^H, B_r^H) . That is,

$$(3.1) \quad \varphi_{s,r}(x, y) = \frac{1}{2\pi\rho_{s,r}} \exp \left\{ -\frac{1}{2\rho_{s,r}^2} (r^{2H}x^2 - 2\mu_{s,r}xy + s^{2H}y^2) \right\},$$

where $\mu_{s,r} = E(B_s^H B_r^H)$ and $\rho_{s,r}^2 = (rs)^{2H} - \mu_{s,r}^2$. Denote the density function of B_s^H by $\varphi_s(x)$. The following Lemmas give some properties and estimates of $\Psi_{s,r,a,b}(x, y)$. The first lemma is a simple calculus exercise.

Lemma 3.1. *Let $G_i \in C^\infty(\mathbb{R})$ have compact supports for $i = 1, 2$. Then we have*

$$\begin{aligned} (3.2) \quad & \int_{\mathbb{R}^2} G'_1(x-a)G'_2(y-b)\varphi_{s,r}(x, y)dx dy \\ & = \int_{\mathbb{R}^2} G'_1(x-a)G'_2(y-b)\Psi_{s,r,a,b}(x, y)dx dy \\ & \quad - G_2(1) \int_{\mathbb{R}} G_1(x-a) \frac{\partial}{\partial x} \varphi_{s,r}(x, b) dx \\ & \quad - G_1(1) \int_{\mathbb{R}} G_2(y-b) \frac{\partial}{\partial y} \varphi_{s,r}(a, y) dy - \varphi_{s,r}(a, b) G_1(1) G_2(1) \end{aligned}$$

for all $r, s > 0$ and $a, b \in \mathbb{R}$, and moreover, if $G_i(1) = 0$ for $i = 1, 2$, we then have

$$(3.3) \quad \int_{\mathbb{R}^2} G'_1(x-a)G'_2(y-b)\varphi_{s,r}(x, y)dx dy = \int_{\mathbb{R}^2} G'_1(x-a)G'_2(y-b)\Psi_{s,r,a,b}(x, y)dx dy$$

for all $r, s > 0$ and $a, b \in \mathbb{R}$.

Lemma 3.2. *For any $x, y, z \in \mathbb{R}$ and $\beta \in [0, 1]$ we have*

$$(3.4) \quad |\varphi_{s,r}(x, y) - \varphi_{s,r}(z, y)| \leq \frac{r^{\beta H}}{\rho_{s,r}^{1+\beta}} |x-z|^\beta e^{-\frac{\beta}{2r^{2H}}y^2}$$

and

$$(3.5) \quad |\varphi_{s,r}(x, y) - \varphi_{s,r}(x, z)| \leq \frac{s^{\beta H}}{\rho_{s,r}^{1+\beta}} |y - z|^\beta e^{-\frac{\beta}{2s^{2H}} x^2}.$$

Proof. We have

$$\begin{aligned} & \left| e^{-\frac{1}{2\rho_{s,r}^2}(r^{2H}x^2 - 2\mu_{s,r}xy + s^{2H}y^2)} - e^{-\frac{1}{2\rho_{s,r}^2}(r^{2H}z^2 - 2\mu_{s,r}zy + s^{2H}y^2)} \right| \\ & \leq \left| e^{-\frac{1}{2\rho_{s,r}^2}(r^{2H}x^2 - 2\mu_{s,r}xy + s^{2H}y^2)} - e^{-\frac{1}{2\rho_{s,r}^2}(r^{2H}z^2 - 2\mu_{s,r}zy + s^{2H}y^2)} \right|^\beta \end{aligned}$$

for all $\beta \in [0, 1]$. It follows from Mean Value Theorem that

$$\begin{aligned} & |\varphi_{s,r}(x, y) - \varphi_{s,r}(z, y)| \\ & \leq \frac{1}{2\pi\rho_{s,r}} \left| e^{-\frac{1}{2\rho_{s,r}^2}(r^{2H}x^2 - 2\mu_{s,r}xy + s^{2H}y^2)} - e^{-\frac{1}{2\rho_{s,r}^2}(r^{2H}z^2 - 2\mu_{s,r}zy + s^{2H}y^2)} \right|^\beta \\ & = \frac{1}{2\pi\rho_{s,r}} \left| (x - z) \left(-\frac{1}{\rho_{s,r}^2} \right) (r^{2H}\xi - \mu_{s,r}y) e^{-\frac{1}{2\rho_{s,r}^2}(r^{2H}\xi^2 - 2\mu_{s,r}\xi y + s^{2H}y^2)} \right|^\beta \\ & = \frac{1}{2\pi\rho_{s,r}} \left| (x - z) \frac{r^{2H}}{\rho_{s,r}^2} \left(\xi - \frac{\mu_{s,r}}{r^{2H}} y \right) e^{-\frac{r^{2H}}{2\rho_{s,r}^2} \left(\xi - \frac{\mu_{s,r}}{r^{2H}} y \right)^2} e^{-\frac{y^2}{2r^{2H}}} \right|^\beta \end{aligned}$$

for some ξ between z and x . Combining this with the fact $|x|e^{-x^2} \leq 1$, we get

$$|\varphi_{s,r}(x, y) - \varphi_{s,r}(z, y)| \leq \frac{r^{\beta H}}{\rho_{s,r}^{1+\beta}} |x - z|^\beta e^{-\frac{\beta}{2r^{2H}} y^2}$$

for all $\beta \in [0, 1]$. Similarly, one can obtain the estimate (3.5). \square

Lemma 3.3. *The estimate*

$$\Lambda_1(s, r, a, b) := \int_a^\infty \int_b^\infty \frac{|\Psi_{s,r,a,b}(x, y)|}{(x - a)(y - b)} dx dy \leq \frac{C_{H,T,\beta} s^{\beta H/2}}{r^{(1+\beta)H} (s - r)^{(1+\beta)H}}$$

holds for all $\beta \in (0, 1)$, $0 < r < s \leq T$ and $a, b \in \mathbb{R}$.

Proof. We have

$$\begin{aligned} & \Lambda_1(s, r, a, b) = \int_a^\infty \int_b^\infty \frac{1}{(x - a)(y - b)} |\Psi_{s,r,a,b}(x, y)| dx dy \\ (3.6) \quad & \leq \int_a^{a+1} dx \int_b^{b+1} \frac{1}{(x - a)(y - b)} |\varphi_{s,r}(x, y) - \varphi_{s,r}(x, b) - \varphi_{s,r}(a, y) + \varphi_{s,r}(a, b)| dy \\ & \quad + \int_{a+1}^\infty dx \int_b^{b+1} \frac{1}{(x - a)(y - b)} |\varphi_{s,r}(x, y) - \varphi_{s,r}(x, b)| dy \\ & \quad + \int_a^{a+1} dx \int_{b+1}^\infty \frac{1}{(x - a)(y - b)} |\varphi_{s,r}(x, y) - \varphi_{s,r}(a, y)| dy \\ & \quad + \int_{a+1}^\infty dx \int_{b+1}^\infty \frac{1}{(x - a)(y - b)} \varphi_{s,r}(x, y) dy \\ & \equiv \Lambda_{11}(s, r, a, b) + \Lambda_{12}(s, r, a, b) + \Lambda_{13}(s, r, a, b) + \Lambda_{14}(s, r, a, b). \end{aligned}$$

Clearly, $\Lambda_{14}(s, r, a, b) \leq 1$ and we have

$$\begin{aligned} \Lambda_{12}(s, r, a, b) + \Lambda_{13}(s, r, a, b) &\leq 2 \frac{s^{\beta H}}{\rho_{s,r}^{1+\beta}} \int_{a+1}^{\infty} dx \int_b^{b+1} \frac{1}{(x-a)(y-b)^{1-\beta}} e^{-\frac{\beta}{2s^{2H}}x^2} dy \\ &\quad + 2 \frac{r^{\beta H}}{\rho_{s,r}^{1+\beta}} \int_a^{a+1} dx \int_{b+1}^{\infty} \frac{1}{(x-a)^{1-\beta}(y-b)} e^{-\frac{\beta}{2r^{2H}}y^2} dy \\ &\leq \frac{C_{H,\beta} s^{(1+\beta)H}}{r^{(1+\beta)H} (s-r)^{(1+\beta)H}} \end{aligned}$$

and $a, b \in \mathbb{R}$ by Lemma 3.2 with $\beta \in (0, 1)$ and Lemma 2.1. In order to estimate $\Lambda_{11}(s, r, a, b)$, by Lemma 3.2 we have

$$(3.7) \quad |\varphi_{s,r}(x, y) - \varphi_{s,r}(x, b) - \varphi_{s,r}(a, y) + \varphi_{s,r}(a, b)| \leq 2 \frac{s^{\beta H}}{\rho_{s,r}^{1+\beta}} |y - b|^\beta$$

and

$$(3.8) \quad |\varphi_{s,r}(x, y) - \varphi_{s,r}(x, b) - \varphi_{s,r}(a, y) + \varphi_{s,r}(a, b)| \leq 2 \frac{r^{\beta H}}{\rho_{s,r}^{1+\beta}} |x - a|^\beta$$

for all $a, b, x, y \in \mathbb{R}$, which give

$$(3.9) \quad |\varphi_{s,r}(x, y) - \varphi_{s,r}(x, b) - \varphi_{s,r}(a, y) + \varphi_{s,r}(a, b)| \leq 2 \frac{(sr)^{\beta H/2}}{\rho_{s,r}^{1+\beta}} |(x-a)(y-b)|^{\beta/2}.$$

It follows from Lemma 2.1 that

$$\begin{aligned} \Lambda_{11}(s, r, a, b) &= \int_a^{a+1} dx \int_b^{b+1} \frac{dy}{(x-a)(y-b)} |\varphi_{s,r}(x, y) - \varphi_{s,r}(x, b) - \varphi_{s,r}(a, y) + \varphi_{s,r}(a, b)| \\ &\leq 2 \frac{(sr)^{\beta H/2}}{\rho_{s,r}^{1+\beta}} \int_a^{a+1} dx \int_b^{b+1} \frac{1}{(x-a)^{1-\frac{\beta}{2}}(y-b)^{1-\frac{\beta}{2}}} dy \\ &\leq C_{H,\beta} \frac{s^{\beta H/2}}{r^{(2+\beta)H/2} (s-r)^{(1+\beta)H}} \end{aligned}$$

for all $\beta \in (0, 1)$ and $a, b \in \mathbb{R}$. This completes the proof. \square

The next proposition shows the process

$$(3.10) \quad \mathcal{C}_t^{+,H}(a) := 2 \left(F_+(B_t^H - a) - F_+(-a) - \int_0^t F'_+(B_s^H - a) \delta B_s^H \right)$$

exists in $L^2(\Omega)$, where

$$(3.11) \quad F_+(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x \log x - x, & \text{if } x > 0. \end{cases}$$

Proposition 3.1. *Let the function F_+ be given as above. Then the random variable*

$$(3.12) \quad \int_0^t F'_+(B_s^H - a) \delta B_s^H$$

exists and

$$\begin{aligned} (3.13) \quad E \left| \int_0^t F'_+(B_s^H - a) \delta B_s^H \right|^2 &= \int_0^t \int_0^t E[F'_+(B_s^H - a) F'_+(B_r^H - a)] \phi(s, r) ds dr \\ &\quad + \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \phi(s, \eta) \phi(r, \xi) \int_a^\infty \int_a^\infty \frac{\Psi_{s,r,a,a}(x, y) dx dy}{(x-a)(y-a)} \end{aligned}$$

for all $t \geq 0$ and $a \in \mathbb{R}$.

By smooth approximation we can obtain the statement. Define the function ζ on \mathbb{R} by

$$(3.14) \quad \zeta(x) := \begin{cases} ce^{\frac{1}{(x-1)^2-1}}, & x \in (0, 2), \\ 0, & \text{otherwise,} \end{cases}$$

where c is a normalizing constant such that $\int_{\mathbb{R}} \zeta(x) dx = 1$. Define the mollifiers

$$(3.15) \quad \zeta_n(x) := n\zeta(nx), \quad n = 2, \dots$$

and the sequence of smooth functions

$$(3.16) \quad G_n(x) := \int_{\mathbb{R}} F'_+(y) \zeta_n(x-y) dy = n \int_{x-\frac{2}{n}}^x F'_+(y) \zeta(n(x-y)) dy$$

$$(3.17) \quad = \int_0^2 F'_+(x - \frac{y}{n}) \zeta(y) dy, \quad n = 2, \dots$$

Then $G_n \in C^\infty(\mathbb{R})$ with compact support for all $n \geq 2$.

Lemma 3.4. *Let the functions $G_n, n \geq 2$ be defined as above. Then we have*

$$|G_n(x)| \leq \psi_1(x) := \begin{cases} 0, & \text{if } x \leq 0, \\ C(1 + |\log x|), & \text{if } x > 0 \end{cases}$$

for all $x \in \mathbb{R}$, and

$$(3.18) \quad G_n(x) \longrightarrow F'_+(x)$$

for all $x \neq 0$, as n tends to infinity.

Proof. Clearly, $G_n(x) = 0$ for $x \leq 0$ and

$$(3.19) \quad G_n(x) = n \int_{(x-\frac{2}{n}) \vee 0}^x \zeta(n(x-y)) \log y dy$$

for $x > 0$.

When $0 < x \leq \frac{2}{n}$, we have

$$\begin{aligned} |G_n(x)| &\leq n \int_0^x \zeta(n(x-y)) |\log y| dy \leq -n \int_0^x \log y dy \\ &= nx(1 - \log x) \leq 2(1 - \log x). \end{aligned}$$

On the other hand, by (3.17) we get

$$\begin{aligned} G_n(x) &= n \int_0^2 \zeta(z) \log(x - \frac{z}{n}) dz = \int_0^2 \zeta(z) \log[x(1 - \frac{z}{nx})] dz \\ &= \log x \int_0^2 \zeta(z) dz + \int_0^2 \zeta(z) \log(1 - \frac{z}{nx}) dz \\ &= \log x + \int_0^2 \zeta(z) \log(1 - \frac{z}{nx}) dz \end{aligned}$$

for $x > \frac{2}{n}$, which gives

$$|G_n(x)| \leq C(1 + |\log x|)$$

for $x > \frac{2}{n}$ since

$$\int_0^2 |\log(1 - \frac{z}{u})| e^{-\frac{1}{1-(1-z)^2}} dz < \infty$$

with $u > 2$.

Finally, the convergence (3.18) follows from $G_n(x) = F'_\alpha(x) = 0$ for $x < 0$ and the next estimate:

$$\begin{aligned}
|G_n(x) - F'_+(x)| &\leq \alpha^{-1} \int_0^2 \zeta(y) \left| F'_+(x - \frac{y}{n}) - F'_+(x) \right| dy \\
&= \alpha^{-1} \int_0^2 \left| \log(x - \frac{y}{n}) - \log x \right| \zeta(y) dy \\
&\leq \alpha^{-1} \int_0^2 \log \left(1 + \frac{y}{nx - y} \right) \zeta(y) dy \\
&\leq \frac{1}{\alpha} \log \left(1 + \frac{2}{nx - 2} \right) \int_0^2 \zeta(y) dy = \frac{1}{\alpha} \log \left(1 + \frac{2}{nx - 2} \right)
\end{aligned}$$

for all $x > \frac{2}{n}$. This completes the proof. \square

Lemma 3.5. *Let the functions $G_n, n \geq 2$ be defined as above. Then we have*

$$|G'_n(x)| \leq \psi_2(x) := \begin{cases} 0, & \text{if } x \leq 0, \\ Cx^{-1}(1 + |\log x|), & \text{if } x > 0 \end{cases}$$

for any $x \in \mathbb{R}$, and

$$(3.20) \quad G'_n(x) \longrightarrow F''_+(x)$$

for all $x \neq 0$, as n tends to infinity.

Proof. Clearly, $G'_n(x) = 0$ for $x \leq 0$, and we have for $x > \frac{2}{n}$,

$$\begin{aligned}
(3.21) \quad G'_n(x) &= \int_0^2 F''_+(x - \frac{y}{n}) \zeta(y) dy = \int_0^2 \zeta(y) \frac{1}{x - \frac{y}{n}} dy \\
&= \int_0^2 \zeta(y) \frac{n}{nx - y} dy = \frac{1}{x} \int_0^2 \zeta(y) \left(1 + \frac{y}{nx - y} \right) dy
\end{aligned}$$

by (3.16). It follows that

$$|G'_n(x)| \leq \frac{1}{x} \int_0^2 \zeta(y) \left(1 + \frac{y}{2 - y} \right) dy \leq \frac{C}{x}$$

for $x > \frac{2}{n}$. On the other hand, for $0 < x \leq \frac{2}{n}$ we have

$$\begin{aligned}
G'_n(x) &= n \int_{\mathbb{R}} F'_+(y) \frac{\partial}{\partial x} \zeta(n(x - y)) dy \\
&= -2n^2 \int_{x - \frac{2}{n}}^x \frac{1 - n(x - y)}{(1 - (1 - n(x - y))^2)^2} \zeta(n(x - y)) F'_+(y) dy.
\end{aligned}$$

Combining this with the fact

$$x^2 e^{-x} \leq 2 \quad (x \geq 0)$$

lead to

$$\begin{aligned}
|G'_n(x)| &\leq 4n^2 \int_0^x |F'_+(y)| |1 - n(x - y)| dy \leq 8n^2 \int_0^x |F'_+(y)| dy \\
&= -8n^2 \int_0^x \log y dy = 8n^2 x (1 - \log x) \leq \frac{32}{x} (1 - \log x)
\end{aligned}$$

for $0 < x \leq \frac{2}{n}$, which gives the estimates of $G'_n(x)$ with $0 < x \leq \frac{2}{n}$.

Finally, by the estimate

$$\int_0^2 \zeta(y) \left(1 + \frac{y}{nx - y} \right) dy \leq \int_0^2 \zeta(y) \left(1 + \frac{y}{2 - y} \right) dy < \infty$$

for all $x > \frac{2}{n}$ and Lebesgue's dominated convergence theorem we have

$$\lim_{u \rightarrow \infty} \int_0^2 \zeta(y) \left(1 + \frac{y}{u-y}\right) dy = 1.$$

Combining this with (3.21), we get the convergence (3.20) since $G'_n(x) = F''_+(x) = 0$ for all $x < 0$. This completes the proof. \square

Lemma 3.6. *Let ψ_2 be defined in Lemma 3.5. Then the estimate*

$$\int_a^\infty \int_b^\infty \psi_2(x-a)\psi_2(y-b)|\Psi_{s,r,a,b}(x,y)|dxdy \leq \frac{C_{H,t}s^{\gamma H/2}}{r^{(1+\gamma)H}(s-r)^{(1+\gamma)H}}$$

holds for all $\gamma \in (0,1)$, $0 < r < s \leq t$ and $a, b \in \mathbb{R}$.

Proof. Similar to Lemma 3.3 one can obtain the estimate since

$$|\log x| \leq C(x^{-\beta} + x^\beta)$$

for all $x > 0$ and all $0 < \beta < 1$. \square

Lemma 3.7. *Let ψ_1 be defined in Lemma 3.4. Then we have*

$$(3.22) \quad \int_{\mathbb{R}} \psi_1(x-a) \left| \frac{\partial}{\partial x} \varphi_{s,r}(x,a) \right| dx \leq C_{H,t,\alpha} (s-r)^{-(1+\alpha)H} r^{-(1+\alpha)H}$$

$$(3.23) \quad \int_{\mathbb{R}} \psi_1(y-a) \left| \frac{\partial}{\partial y} \varphi_{s,r}(a,y) \right| dy \leq C_{H,t,\alpha} (s-r)^{-(1+\alpha)H} r^{-(1+2\alpha)H}$$

for all $a \in \mathbb{R}$, $0 < r < s \leq t$ and $1-H < \alpha < 1$.

Proof. Given $a \in \mathbb{R}$ and $0 < r < s \leq t$. Make the substitution $\frac{r^H}{\rho}(x - \frac{\mu}{r^{2H}}a) = y$. Then

$$\begin{aligned} & \int_a^\infty |\log(x-a)| \left| \frac{\partial}{\partial x} \varphi_{s,r}(x,a) \right| dx \\ &= \frac{r^H}{\rho} \int_a^\infty |\log(x-a)| \left| \frac{r^H}{\rho} \left(x - \frac{\mu}{r^{2H}}a \right) \right| \varphi_{s,r}(x,a) dx \\ &= \frac{1}{2\pi\rho} e^{-\frac{a^2}{2r^{2H}}} \int_{-\frac{\mu-r^{2H}}{\rho r^H}a}^\infty \left| \log \left\{ \frac{\rho}{r^H} \left(y + \frac{\mu-r^{2H}}{\rho r^H}a \right) \right\} \right| |y| e^{-\frac{1}{2}y^2} dy \\ &\leq \frac{1}{2\pi\rho} \left(\int_{-\frac{\mu-r^{2H}}{\rho r^H}a}^\infty \left| \log \frac{\rho}{r^H} \right| |y| e^{-\frac{1}{2}y^2} dy \right. \\ &\quad \left. + e^{-\frac{a^2}{2r^{2H}}} \int_{-\frac{\mu-r^{2H}}{\rho r^H}a}^\infty \left| \log \left(y + \frac{\mu-r^{2H}}{\rho r^H}a \right) \right| |y| e^{-\frac{1}{2}y^2} dy \right) \\ &\equiv \frac{1}{\rho} (\Delta_1(s,r,a) + \Delta_2(s,r,a)). \end{aligned}$$

By Lemma 2.1 and the fact $|\log x| \leq x + x^{-\alpha}$ for all $x > 0$ and $\alpha \in (0,1)$ we see that

$$\Delta_1(s,r,a) \leq \left(\left(\frac{\rho}{r^H} \right)^{-\alpha} + \frac{\rho}{r^H} \right) \int_{-\frac{\mu-r^{2H}}{\rho r^H}a}^\infty |y| e^{-\frac{1}{2}y^2} dy \leq \frac{r^{H\alpha}}{\rho^\alpha} + \frac{\rho}{r^H} \leq \frac{C_{H,t,\alpha}}{(s-r)^{H\alpha}}$$

and

$$\begin{aligned}
\Delta_2(s, r, a) &\leq e^{-\frac{a^2}{2r^{2H}}} \int_{-\frac{\mu-r^{2H}}{\rho r^H}a}^{\infty} \left(y + \frac{\mu-r^{2H}}{\rho r^H}a \right) |y| e^{-\frac{1}{2}y^2} dy \\
&\quad + e^{-\frac{a^2}{2r^{2H}}} \int_{-\frac{\mu-r^{2H}}{\rho r^H}a}^{\infty} \left(y + \frac{\mu-r^{2H}}{\rho r^H}a \right)^{\alpha-1} |y| e^{-\frac{1}{2}y^2} dy \\
&\equiv \Delta_{21}(s, r, a) + \Delta_{22}(s, r, a).
\end{aligned}$$

Now, let us estimate $\Delta_{21}(s, r, a)$ and $\Delta_{22}(s, r, a)$. We have

$$\begin{aligned}
\Delta_{21}(s, r, a) &\leq \int_{-\frac{\mu-r^{2H}}{\rho r^H}a}^{\infty} |y|^2 e^{-\frac{1}{2}y^2} dy + e^{-\frac{a^2}{2r^{2H}}} \int_{-\frac{\mu-r^{2H}}{\rho r^H}a}^{\infty} \frac{\mu-r^{2H}}{\rho r^H} |a| |y| e^{-\frac{1}{2}y^2} dy \\
&\leq \int_{-\infty}^{\infty} |y|^2 e^{-\frac{1}{2}y^2} dy + \left(\frac{|a|}{r^H} e^{-\frac{a^2}{2r^{2H}}} \right) \left(\frac{\mu-r^{2H}}{\rho} \right) \int_{-\infty}^{\infty} |y| e^{-\frac{1}{2}y^2} dy \\
&\leq \sqrt{2\pi} \left(1 + \frac{\mu-r^{2H}}{\rho} \right)
\end{aligned}$$

by the fact $|y|e^{-\frac{1}{2}y^2} \leq 1$. On the other hand, we have also

$$\begin{aligned}
&\Delta_{22}(s, r, a) 1_{\{a \geq 0\}} \\
&\leq e^{-\frac{a^2}{2r^{2H}}} \left(\int_{-\frac{\mu-r^{2H}}{\rho r^H}a}^0 \left(y + \frac{\mu-r^{2H}}{\rho r^H}a \right)^{\alpha-1} |y| e^{-\frac{1}{2}y^2} dy + \int_0^{\infty} y^{\alpha+1} e^{-\frac{1}{2}y^2} dy \right) 1_{\{a \geq 0\}} \\
&\leq \frac{1}{\alpha} e^{-\frac{a^2}{2r^{2H}}} \left(\frac{\mu-r^{2H}}{\rho r^H}a \right)^{\alpha} 1_{\{a \geq 0\}} + C_{\alpha} 1_{\{a \geq 0\}} \\
&\leq C_{\alpha} \frac{(\mu-r^{2H})^{\alpha}}{\rho^{\alpha}} 1_{\{a \geq 0\}} + C_{\alpha} 1_{\{a \geq 0\}}
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{22}(s, r, a) 1_{\{a < 0\}} &\leq 1_{\{a < 0\}} \int_{-\frac{\mu-r^{2H}}{\rho r^H}a}^{\frac{\mu-r^{2H}}{\rho r^H}(1-a)} \left(y + \frac{\mu-r^{2H}}{\rho r^H}a \right)^{\alpha-1} |y| e^{-\frac{1}{2}y^2} dy dy \\
&\quad + 1_{\{a < 0\}} \int_{\frac{\mu-r^{2H}}{\rho r^H}(1-a)}^{\infty} \left(\frac{\mu-r^{2H}}{\rho r^H} \right)^{\alpha-1} |y| e^{-\frac{1}{2}y^2} dy dy \\
&\leq \alpha^{-1} \frac{(\mu-r^{2H})^{\alpha}}{(\rho r^H)^{\alpha}} 1_{\{a < 0\}} + \frac{(\mu-r^{2H})^{\alpha-1}}{(\rho r^H)^{\alpha-1}} 1_{\{a < 0\}}
\end{aligned}$$

by the fact $|y|^{\alpha} e^{-\frac{1}{2}y^2} \leq 1$ with $0 \leq \alpha \leq 1$. It follows from Lemma 2.1 and Lemma 2.2 that

$$\begin{aligned}
\Delta_2(s, r, a) &= \Delta_{21}(s, r, a) + \Delta_{22}(s, r, a) \\
&\leq C_{\alpha} \left(1 + \frac{\mu-r^{2H}}{\rho} + \frac{(\mu-r^{2H})^{\alpha}}{\rho^{\alpha}} + \frac{(\mu-r^{2H})^{\alpha}}{(\rho r^H)^{\alpha}} + \frac{(\mu-r^{2H})^{\alpha-1}}{(\rho r^H)^{\alpha-1}} \right) \\
&\leq C_{H, \alpha, t} (s-r)^{-(1-H)(1-\alpha)} r^{-\alpha H}
\end{aligned}$$

for all $0 < r < s \leq t$. Combining this with Lemma 2.1, we have

$$\begin{aligned}
& \int_{\mathbb{R}} \psi_1(x-a) \left| \frac{\partial}{\partial x} \varphi_{s,r}(x,a) \right| dx \\
&= \int_{\mathbb{R}} \left| \frac{\partial}{\partial x} \varphi_{s,r}(x,a) \right| dx + \int_a^\infty |\log(x-a)| \left| \frac{\partial}{\partial x} \varphi_{s,r}(x,a) \right| dx \\
&\leq \frac{C_{H,\alpha,t}}{\rho} (1 + \Delta_1(s,r,a) + \Delta_2(s,r,a)) \\
&\leq \frac{C_{H,\alpha,t}}{\rho} \left(1 + (s-r)^{-H\alpha} + r^{-\alpha H} (s-r)^{-(1-H)(1-\alpha)} \right) \\
&\leq C_{H,\alpha,t} (s-r)^{-(1+\alpha)H} r^{-(1+\alpha)H}
\end{aligned}$$

for all $0 < r < s \leq t$ and $1-H < \alpha \leq 1$. Similarly, we can obtain the estimate (3.23). \square

Now, we can prove Proposition 3.1.

Proof of Proposition 3.1. Let $G_n, n \geq 2$ be defined in (3.16). Then

$$E \left| G_n(B_s^H - a) - F'_+(B_s^H - a) \right|^2 \longrightarrow 0 \quad (n \rightarrow \infty)$$

for all $s \geq 0$ and $a \in \mathbb{R}$ by Lemma 3.4, Lebesgue's dominated convergence theorem and the next estimate:

$$(3.24) \quad E[\psi_1(B_s^H - a)^2] = \int_a^\infty (1 + |\log(x-a)|)^2 \varphi_s(x) dx < \infty$$

for all $s \geq 0$ and $a \in \mathbb{R}$. Thus, it is sufficient to show that the sequence

$$Y_t^H(n) := \int_0^t G_n(B_s^H - a) \delta B_s^H, \quad n \geq 2$$

is a Cauchy sequence in $L^2(\Omega)$. Denote $\tilde{G}_{n,m} = G_n - G_m$ for all $n, m \geq 2$. Then $Y_t^H(n)$ is a Cauchy sequence in $L^2(\Omega)$ if and only if

$$\begin{aligned}
E \left| Y_t^H(n) - Y_t^H(m) \right|^2 &= E \left| \int_0^t \tilde{G}_{n,m}(B_s^H - a) \delta B_s^H \right|^2 \\
&= \int_0^t \int_0^t E \tilde{G}_{n,m}(B_s^H - a) \tilde{G}_{n,m}(B_r^H - a) \phi(s,r) dr ds \\
&\quad + \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \phi(s,\eta) \phi(r,\xi) \\
&\quad \cdot E \left[\tilde{G}'_{n,m}(B_s^H - a) \tilde{G}'_{n,m}(B_r^H - a) \right] \\
&\equiv \Lambda_{n,m}(1) + \Lambda_{n,m}(2) \longrightarrow 0
\end{aligned}$$

as $n, m \rightarrow \infty$.

On the one hand, we have

$$|\tilde{G}_{n,m}(x)| \leq C \psi_1(x) \quad (x \in \mathbb{R})$$

and $\tilde{G}_{n,m}(x) \rightarrow 0$ for all $x \in \mathbb{R}$, as n, m tends to infinity, by Lemma 3.4. Accrediting with the estimate

$$\begin{aligned}
\Lambda_2(s,r,a,a) &:= \int_0^t \int_0^t E[\psi_1(B_s^H - a) \psi_1(B_r^H - a)] \phi(s,r) ds dr \\
(3.25) \quad &= \int_0^t \int_0^t \phi(s,r) ds dr \int_a^\infty \int_a^\infty (1 + |\log(x-a)|) \\
&\quad \cdot (1 + |\log(y-a)|) \varphi_{s,r}(x,y) dx dy < \infty
\end{aligned}$$

and Lebesgue's dominated convergence theorem, we give the convergence

$$(3.26) \quad \Lambda_{n,m}(1) = \int_0^t \int_0^s \phi(s, r) dr ds \int_{\mathbb{R}^2} \tilde{G}_{n,m}(x-a) \tilde{G}_{n,m}(y-a) \varphi_{s,r}(x, y) dx dy \longrightarrow 0$$

for $a \in \mathbb{R}$, as n, m tend to infinity.

On the other hand, by Lemma 3.1 we have

$$(3.27) \quad \begin{aligned} \Lambda_{n,m}(2) &= \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \phi(s, \eta) \phi(r, \xi) \\ &\quad \cdot \int_{\mathbb{R}^2} \tilde{G}'_{n,m}(x-a) \tilde{G}'_{n,m}(y-a) \Psi_{s,r,a,a}(x, y) dx dy \\ &\quad + \int_0^t ds \int_0^t dr \int_0^s d\xi \int_0^r d\eta \phi(s, \eta) \phi(r, \xi) \Theta_{n,m}(s, r, a, a) \end{aligned}$$

for all $n, m, t \geq 0$ and $a \in \mathbb{R}$, where

$$\begin{aligned} \Theta_{n,m}(s, r, a, b) &= -\tilde{G}_{n,m}(1) \int_{\mathbb{R}} \tilde{G}_{n,m}(x-a) \frac{\partial}{\partial x} \varphi_{s,r}(x, b) dx \\ &\quad - \tilde{G}_{n,m}(1) \int_{\mathbb{R}} \tilde{G}_{n,m}(y-b) \frac{\partial}{\partial y} \varphi_{s,r}(a, y) dy - \varphi_{s,r}(a, b) \left(\tilde{G}_{n,m}(1) \right)^2. \end{aligned}$$

Noting that

$$\int_0^r |s - \xi|^{2H-2} d\xi = \frac{1}{2H-1} (s^{2H-1} + |s-r|^{2H-1} \text{sign}(r-s)),$$

we get

$$(3.28) \quad \int_0^s d\xi \int_0^r |r - \xi|^{2H-2} |s - \eta|^{2H-2} d\eta \leq \frac{2}{(2H-1)^2} r^{2H-1} s^{2H-1}.$$

It follows from Lemma 3.4 and Lemma 3.7 with $1-H < \alpha < \frac{1-H}{H} \wedge \frac{1}{2}$ that

$$(3.29) \quad \begin{aligned} &\int_0^t ds \int_0^t dr \int_0^s d\xi \int_0^r d\eta \phi(s, \eta) \phi(r, \xi) |\Theta_{n,m}(s, r, a, a)| \\ &\leq C_{H,t} \tilde{G}_{n,m}(1) \int_0^t ds \int_0^t dr \int_0^s d\xi \int_0^r d\eta \frac{\phi(s, \eta) \phi(r, \xi)}{|s-r|^{H(1+\alpha)} (s \wedge r)^{(1+\alpha)H}} \\ &\quad + C_{H,t} \tilde{G}_{n,m}(1) \int_0^t ds \int_0^t dr \int_0^s d\xi \int_0^r d\eta \frac{\phi(s, \eta) \phi(r, \xi)}{|s-r|^{H(1+\alpha)} (s \wedge r)^{(1+2\alpha)H}} \\ &\quad + \left(\tilde{G}_{n,m}(1) \right)^2 \int_0^t ds \int_0^t dr \int_0^s d\xi \int_0^r d\eta \phi(s, \eta) \phi(r, \xi) \frac{1}{\rho} \\ &\leq C_{H,t} \tilde{G}_{n,m}(1) \left(1 + \tilde{G}_{n,m}(1) \right) \longrightarrow 0 \quad (n, m \rightarrow \infty) \end{aligned}$$

for all $t \geq 0$ and $a \in \mathbb{R}$. Moreover, (3.28) and Lemma 3.6 imply that

$$(3.30) \quad \begin{aligned} &\int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \phi(s, \eta) \phi(r, \xi) \\ &\quad \cdot \int_{\mathbb{R}^2} \psi_2(x-a) \psi_2(y-a) \Psi_{s,r,a,a}(x, y) dx dy < \infty \end{aligned}$$

for all $t \geq 0$ and $a \in \mathbb{R}$. Combining this with (3.27), (3.29), Lemma 3.5 and Lebesgue's dominated convergence theorem that the convergence, we have

$$\Lambda_{n,m}(2) \longrightarrow 0,$$

as n, m tend to infinity. Thus, we have showed that $Y_t^H(n), n = 1, 2, \dots$ is a Cauchy sequence in $L^2(\Omega)$ and the process

$$\lim_{n \rightarrow \infty} \int_0^t G_n(B_s^H - a) \delta B_s^H = \int_0^t F'_+(B_s^H - a) \delta B_s^H, \quad t \geq 0$$

exists in $L^2(\Omega)$.

Denote

$$\begin{aligned} \tilde{\Theta}_n(s, r, a, b) &:= -G_n(1) \int_{\mathbb{R}} G_n(x - a) \frac{\partial}{\partial x} \varphi_{s,r}(x, b) dx \\ &\quad - G_n(1) \int_{\mathbb{R}} G_n(y - b) \frac{\partial}{\partial y} \varphi_{s,r}(a, y) dy - \varphi_{s,r}(a, b) G_n(1) G_n(1) \end{aligned}$$

for all $a, b \in \mathbb{R}$ and $0 < r < s$. Then, for all $t \geq 0$ and $a \in \mathbb{R}$ we have

$$\begin{aligned} E \left| \int_0^t G_n(B_s^H - a) \delta B_s^H \right|^2 &= \int_0^t \int_0^t EG_n(B_s^H - a) G_n(B_r^H - a) \phi(s, r) dr ds \\ &\quad + \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \phi(s, \eta) \phi(r, \xi) \\ &\quad \cdot E [G'_n(B_s^H - a) G'_n(B_r^H - a)] \\ &= \int_0^t \int_0^t EG_n(B_s^H - a) G_n(B_r^H - a) \phi(s, r) dr ds \\ &\quad + \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \phi(s, \eta) \phi(r, \xi) \\ &\quad \cdot \int_{\mathbb{R}^2} G'_n(x - a) G'_n(y - a) \Psi_{s,r,a,a}(x, y) dx dy \\ &\quad + \int_0^t ds \int_0^t dr \int_0^s d\xi \int_0^r d\eta \phi(s, \eta) \phi(r, \xi) \tilde{\Theta}_n(s, r, a, a) \end{aligned}$$

by Lemma 3.2. Notice that

$$\int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \phi(s, \eta) \phi(r, \xi) \tilde{\Theta}_n(s, r, a, a) \longrightarrow 0,$$

as n tends to infinity, by Lemma 3.4, Lemma 3.7 and (3.28). We introduce the identity (3.13) by taking the limit in $L^2(\Omega)$ and the proposition follows. \square

Finally, by considering the function on \mathbb{R}^2

$$\begin{aligned} \tilde{\Psi}_{s,r,a,b}(x, y) &:= \varphi_{s,r}(x, y) - \varphi_{s,r}(x, b) \theta(y - 1 - b) \\ &\quad - \varphi_{s,r}(a, y) \theta(x - 1 - a) + \varphi_{s,r}(a, b) \theta(x - 1 - a) \theta(y - 1 - b) \end{aligned}$$

with $s, r > 0$ and $a, b \in \mathbb{R}$, and in a same way as proof of Proposition 3.1, we can show that the integral

$$\int_0^t F'_-(B_s^H - a) \delta B_s^H, \quad t \geq 0$$

and the process

$$(3.31) \quad C_t^{-,H}(a) := 2 \left(F_-(B_t^H - a) - F_-(-a) - \int_0^t F'_-(B_s^H - a) \delta B_s^H \right), \quad t \geq 0$$

exist in $L^2(\Omega)$ for all $a \in \mathbb{R}$, where

$$(3.32) \quad F_-(x) = \begin{cases} 0, & \text{if } x \geq 0, \\ x \log(-x) - x, & \text{if } x < 0. \end{cases}$$

Proposition 3.2. *For all $a \in \mathbb{R}$ the integral*

$$(3.33) \quad X_t^H(a) := \int_0^t \log |B_s^H - a| \delta B_s^H,$$

and

$$(3.34) \quad E \left| \int_0^t \log |B_s^H - a| \delta B_s^H \right|^2 = \int_0^t \int_0^t E[\log |B_s^H - a| \log |B_r^H - a|] \phi(s, r) ds dr \\ + \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \phi(s, \eta) \phi(r, \xi) \int_{\mathbb{R}^2} \frac{\Psi_{s,r,a,a}(x, y) dx dy}{(x-a)(y-a)}$$

for all $t \geq 0$ and $a \in \mathbb{R}$ and the process

$$(3.35) \quad \mathcal{C}_t^H(a) := 2 \left(F(B_t^H - a) - F(-a) - \int_0^t \log |B_s^H - a| \delta B_s^H \right), \quad t \geq 0$$

are well defined, where $F(x) = x \log |x| - x$

Proof. Clearly, $F'(x) = \log |x|$, and the proposition follows from $F' = F'_+ + F'_-$. \square

4. A REPRESENTATION OF THE FUNCTIONAL $\mathcal{C}^H(a)$

In this section we will consider the representation of the functionals $\mathcal{C}^{+,H}(a)$, $\mathcal{C}^{-,H}(a)$ and $\mathcal{C}^H(a)$, which point out that $\frac{1}{\pi} \mathcal{C}_t^H(\cdot)$ is the Hilbert transform of weighted local time $\mathcal{L}^H(\cdot, t)$.

Lemma 4.1. *For any $0 < \varepsilon < 1$, $0 < r < s \leq t$ and $\beta \in (0, 1)$ we have*

$$(4.1) \quad \Lambda_3(s, r, a) := \int_a^{a+\varepsilon} \int_a^{a+\varepsilon} \left(\log(x-a) - \left(\frac{1}{\varepsilon} \log \varepsilon\right)(x-a) \right) \\ \cdot \left(\log(y-a) - \left(\frac{1}{\varepsilon} \log \varepsilon\right)(y-a) \right) \varphi_{s,r}(x, y) dx dy \\ \leq C_H (sr)^{-H/2} \varepsilon^H$$

and

$$(4.2) \quad \Lambda_4(s, r, a) := \int_a^{a+\varepsilon} \int_a^{a+\varepsilon} \left(\frac{1}{x-a} - \frac{1}{\varepsilon} \log \varepsilon \right) \left(\frac{1}{y-a} - \frac{1}{\varepsilon} \log \varepsilon \right) |\Psi_{s,r,a,a}(x, y)| dx dy \\ \leq C_{H,t,\beta} \frac{s^{\beta H/2}}{r^{(1+\frac{\beta}{2})H} (s-r)^{(1+\beta)H}} \varepsilon^\beta (1 + \log^2 \varepsilon).$$

Proof. The estimate (4.1) is clear. In order to prove (4.2), we have

$$\int_a^{a+\varepsilon} \int_a^{a+\varepsilon} \left(\frac{1}{x-a} - \frac{1}{\varepsilon} \log \varepsilon \right) \left(\frac{1}{y-a} - \frac{1}{\varepsilon} \log \varepsilon \right) \\ \cdot [(x-a)(y-a)]^{\beta/2} dx dy \leq C_\beta \varepsilon^\beta (1 + \log^2 \varepsilon)$$

for all $\beta \in (0, 1)$, which gives

$$\int_a^{a+\varepsilon} \int_a^{a+\varepsilon} \left(\frac{1}{x-a} - \frac{1}{\varepsilon} \log \varepsilon \right) \left(\frac{1}{y-a} - \frac{1}{\varepsilon} \log \varepsilon \right) |\Psi_{s,r,a,a}(x, y)| dx dy \\ = \int_a^{a+\varepsilon} \int_a^{a+\varepsilon} \left(\frac{1}{x-a} - \frac{1}{\varepsilon} \log \varepsilon \right) \left(\frac{1}{y-a} - \frac{1}{\varepsilon} \log \varepsilon \right) \\ \cdot |\varphi_{s,r}(x, y) - \varphi_{s,r}(x, b) - \varphi_{s,r}(a, y) + \varphi_{s,r}(a, b)| dx dy \\ \leq C_{H,t,\beta} \frac{s^{\beta H/2}}{r^{(1+\frac{\beta}{2})H} (s-r)^{(1+\beta)H}} \varepsilon^\beta (1 + \log^2 \varepsilon)$$

by (3.9) and Lemma 2.1. This completes the proof. \square

Lemma 4.2. *Let $\frac{1}{2} < H < 1$ and $M > 0$. We then have*

$$(4.3) \quad E \left| \mathcal{L}^H(b, t) - \mathcal{L}^H(a, t) \right|^2 \leq C_{H, \alpha, t, M} |b - a|^\alpha$$

for all $0 < \alpha < \frac{1-H}{H}$, $t \geq 0$ and $a, b \in [-M, M]$.

The lemma is a direct consequence of Hölder continuity of $x \mapsto \mathcal{L}^H(x, t)$. Here, we shall use other method to prove it.

Proof of Lemma 4.2. Without loss of generality we may assume that $0 < a < b$. Define the function $f_{a,b}(x) = 1_{(a,b]}(x)$ and denote

$$\tilde{B}_t^H(x) := \int_0^t 1_{\{B_s^H > x\}} \delta B_s^H$$

and

$$\psi_t(x) := (B_t^H - x)^+ - (-x)^+$$

for all $x \in \mathbb{R}$. Then the function $x \mapsto \psi_t(x)$ is Lipschitz continuous with Lipschitz constant 2, and we have

$$|\psi_t(x) - \psi_t(y)| \leq 2|x - y|$$

for all $x, y \in \mathbb{R}$ and

$$\mathcal{L}^H(x, t) = 2 \left(\psi_t(x) - \tilde{B}_t^H(x) \right).$$

by Tanaka formula, which deduces

$$E \left| \mathcal{L}^H(b, t) - \mathcal{L}^H(a, t) \right|^2 \leq 8(b - a)^2 + 4E \left| \tilde{B}_t^H(b) - \tilde{B}_t^H(a) \right|^2$$

for all $t \geq 0$.

On the other hand, similar to the proof of (3.13) by approximating the function $f(x) = 1_{(a,b]}(x)$ by smooth functions we can obtain

$$(4.4) \quad \begin{aligned} E \left(\tilde{B}_t^H(b) - \tilde{B}_t^H(a) \right)^2 &= E \left(\int_0^t f_{a,b}(B_s^H) \delta B_s^H \right)^2 \\ &= \int_0^t \int_0^t E [f_{a,b}(B_s^H) f_{a,b}(B_r^H)] \phi(s, r) ds dr \\ &\quad + \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \Lambda_5(s, r, a, b) \phi(s, \eta) \phi(r, \xi) \\ &\equiv G_1(a, b) + G_2(a, b) \end{aligned}$$

for all $\frac{1}{2} < H < 1$, where

$$\Lambda_5(s, r, a, b) = \varphi_{s,r}(a, a) - \varphi_{s,r}(a, b) - \varphi_{s,r}(b, a) + \varphi_{s,r}(b, b).$$

For the first term, we have

$$\begin{aligned} E [f_{a,b}(B_s^H) f_{a,b}(B_r^H)] &= \int_a^b \int_a^b \frac{1}{2\pi \rho_{s,r}} \exp \left(-\frac{1}{2\rho_{s,r}^2} (r^{2H} x^2 - 2\mu_{s,r} x y + s^{2H} y^2) \right) dx dy \\ &= \frac{1}{2\pi} \int_{\frac{a}{r^H}}^{\frac{b}{r^H}} e^{-\frac{1}{2}x^2} dx \int_{\frac{ar^H - \mu_{s,r}x}{\rho_{s,r}}}^{\frac{br^H - \mu_{s,r}x}{\rho_{s,r}}} e^{-\frac{1}{2}y^2} dy \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{\frac{a}{r^H}}^{\frac{b}{r^H}} e^{-\frac{1}{2}x^2} dx \left(\frac{1}{\sqrt{2\pi}} \int_{\frac{ar^H - \mu_{s,r}x}{\rho_{s,r}}}^{\frac{br^H - \mu_{s,r}x}{\rho_{s,r}}} e^{-\frac{1}{2}y^2} dy \right)^\beta \\ &\leq \left(\frac{r^H(b-a)}{\rho_{s,r}} \right)^\beta \int_{\frac{a}{r^H}}^{\frac{b}{r^H}} e^{-\frac{1}{2}x^2} dx \leq \frac{r^{(\alpha-1)H}}{\rho_{s,r}^\beta} (b-a)^{1+\beta}, \end{aligned}$$

for all $s, r > 0$ and $\beta \in (0, 1)$. It follows from Lemma 2.1 that

$$(4.5) \quad \begin{aligned} G_1(a, b) &= \int_0^t \int_0^t E[f_{a,b}(B_s^H) f_{a,b}(B_r^H)] \phi(s, r) ds dr \\ &\leq C_{H,\beta} t^{H(1-\beta)} (b-a)^{1+\beta} \end{aligned}$$

for all $\frac{1}{2} < H < 1$ and $0 \leq \beta < \frac{2H-1}{H}$.

For the second term, we have also by (3.8) and Lemma 2.1

$$\begin{aligned} G_2(a, b) &= \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \Lambda_5(a, b, s, r) \phi(s, \eta) \phi(r, \xi) \\ &\leq C_H (b-a)^\alpha \int_0^t \int_0^t \frac{(sr)^{2H-1} r^{\alpha H}}{\rho_{s,r}^{1+\alpha}} dr ds \leq C_{H,\alpha} (b-a)^\alpha t^{H(2-\alpha)} \end{aligned}$$

for all $0 < \alpha < \frac{1-H}{H}$, and the lemma follows. \square

The main object of this section is to prove the following theorem.

Theorem 4.1. *The convergence*

$$(4.6) \quad C_t^{+,H}(a) = \lim_{\varepsilon \downarrow 0} \left\{ (\log \varepsilon) \mathcal{L}^H(a, t) + \int_0^t 1_{\{B_s^H - a \geq \varepsilon\}} \frac{2H s^{2H-1}}{B_s^H - a} ds \right\}$$

holds in $L^2(\Omega)$ for all $t \geq 0$.

Proof. Let $t \geq 0$ and $a \in \mathbb{R}$. We split the proof in three steps.

Step I. Define the function F_ε as follows

$$F_\varepsilon(x) = \begin{cases} 0, & x \leq 0, \\ \frac{1}{2\varepsilon}(x^2 \log \varepsilon), & 0 < x \leq \varepsilon, \\ \varepsilon - \frac{1}{2}(\varepsilon \log \varepsilon) + x \log x - x, & x > \varepsilon. \end{cases}$$

Then $F_\varepsilon \in C^1(\mathbb{R})$, and

$$F'_\varepsilon(x) = \begin{cases} 0, & x \leq 0, \\ \frac{1}{\varepsilon}(x \log \varepsilon), & 0 < x \leq \varepsilon, \\ \log x, & x > \varepsilon, \end{cases} \quad F''_\varepsilon(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{\varepsilon} \log \varepsilon, & 0 < x < \varepsilon, \\ \frac{1}{x}, & x > \varepsilon \end{cases}$$

for all $\varepsilon \in (0, 1)$. We shall show that the Itô formula

$$(4.7) \quad \begin{aligned} F_\varepsilon(B_t^H - a) - F_\varepsilon(-a) - \int_0^t F'_\varepsilon(B_s^H - a) \delta B_s^H \\ = \frac{\log \varepsilon}{\varepsilon} \int_0^t 1_{\{0 \leq B_s^H - a < \varepsilon\}} H s^{2H-1} ds + \int_0^t 1_{\{B_s^H - a \geq \varepsilon\}} \frac{H s^{2H-1}}{B_s^H - a} ds \end{aligned}$$

holds for $\varepsilon \in (0, 1)$. Define the sequence of smooth functions

$$(4.8) \quad f_{n,\varepsilon}(x) := \int_{\mathbb{R}} F_\varepsilon(x-y) \zeta_n(y) dy = \int_0^2 F_\varepsilon(x - \frac{y}{n}) \zeta(y) dy, \quad n = 1, 2, \dots$$

for all $\varepsilon \in (0, 1)$, where ζ is defined by (3.14) and $\zeta_n(x) = n\zeta(nx)$. Then $f_{n,\varepsilon} \in C_0^\infty(\mathbb{R})$ and

$$f_{n,\varepsilon}(B_t^H - a) = f_{n,\varepsilon}(-a) + \int_0^t f'_{n,\varepsilon}(B_s^H - a) \delta B_s^H + H \int_0^t f''_{n,\varepsilon}(B_s^H - a) s^{2H-1} ds$$

for all $n = 1, 2, \dots$. Notice that

$$f_{n,\varepsilon}(x) \longrightarrow F_\varepsilon(x), \quad f'_{n,\varepsilon}(x) \longrightarrow F'_\varepsilon(x)$$

as $n \rightarrow \infty$, uniformly in \mathbb{R} , and

$$|f''_{n,\varepsilon}(x)| \leq \frac{1}{\varepsilon} |\log \varepsilon|, \quad \forall x \in \mathbb{R},$$

and $f''_{n,\varepsilon}(x) \rightarrow F''_\varepsilon(x)$ pointwise (besides 0 and ε), as $n \rightarrow \infty$ by Lebesgue's dominated convergence theorem. We get

$$\begin{aligned} \int_0^t f'_{n,\varepsilon}(B_s^H - a) \delta B_s^H &= f_{n,\varepsilon}(B_t^H - a) - f_{n,\varepsilon}(-a) - H \int_0^t f''_{n,\varepsilon}(B_s^H - a) s^{2H-1} ds \\ &\longrightarrow F_\varepsilon(B_t^H - a) - F_\varepsilon(-a) - H \int_0^t F''_\varepsilon(B_s^H - a) s^{2H-1} ds \quad \text{in } L^2(\Omega) \\ &= F_\varepsilon(B_t^H - a) - F_\varepsilon(-a) - H\varepsilon^{-1} \int_0^t 1_{\{0 < B_s^H - a < \varepsilon\}} s^{2H-1} ds \quad \text{a.s.,} \end{aligned}$$

as $n \rightarrow \infty$, which implies that Itô's formula

$$(4.9) \quad F_\varepsilon(B_t^H - a) = F_\varepsilon(-a) + \int_0^t F'_\varepsilon(B_s^H - a) \delta B_s^H + H \int_0^t F''_\varepsilon(B_s^H - a) s^{2H-1} ds$$

holds for all $\varepsilon \in (0, 1)$. This gives (4.7).

Step II. We show that the limit

$$(4.10) \quad \lim_{\varepsilon \downarrow 0} \left\{ F_\varepsilon(B_t^H - a) - F_\varepsilon(-a) - \int_0^t F'_\varepsilon(B_s^H - a) \delta B_s^H \right\}$$

exists in $L^2(\Omega)$, and is equal to

$$\frac{1}{2} \mathcal{C}_t^{+,H}(a) = F_+(B_t^H - a) - F_+(-a) - \int_0^t F'_+(B_s^H - a) \delta B_s^H,$$

where F_+ is given by (3.11). We have

$$\begin{aligned} (4.11) \quad E \left| \frac{1}{2} \mathcal{C}_t^{+,H}(a) + F_\varepsilon(-a) - F_\varepsilon(B_t^H - a) + \int_0^t F'_\varepsilon(B_s^H - a) \delta B_s^H \right|^2 \\ \leq 3E \left| F(B_t^H - a) - F_\varepsilon(B_t^H - a) \right|^2 + 3|F_\varepsilon(-a) - F_+(-a)|^2 \\ + 3E \left| \int_0^t [F'(B_s^H - a) - F'_\varepsilon(B_s^H - a)] \delta B_s^H \right|^2. \end{aligned}$$

The first and second term of the right-hand side in (4.11) tends to 0 as $\varepsilon \rightarrow 0$ because

$$|F_+(x) - F_\varepsilon(x)| \leq \varepsilon - \frac{1}{2} \varepsilon \log \varepsilon$$

for all $\varepsilon \in (0, 1)$. To estimate the third term, we consider the approximation of the function F'_ε as follows

$$\widehat{G}_{n,\varepsilon}(x) = \int_{\mathbb{R}} F'_\varepsilon(y) \zeta_n(x - y) dy, \quad n \geq 2$$

for all $\varepsilon \in (0, 1)$, where $\zeta_n, n \geq 2$ is given by (3.15). Then $G_{n,\varepsilon}, n \geq 2$ are smooth functions with compact supports. Denote

$$G_{n,\varepsilon}(x) := G_n(x) - \widehat{G}_{n,\varepsilon}(x)$$

for $x \in \mathbb{R}$, where G_n is defined by (3.16). Similar to proofs of Lemma (3.4) and Lemma (3.5), we can obtain the next statements:

$$|G_{n,\varepsilon}(x)| \leq C\psi_1(x), \quad |G'_{n,\varepsilon}(x)| \leq C\psi_2(x)$$

for all $x \in \mathbb{R}, \varepsilon \in (0, 1)$ and

$$G_{n,\varepsilon}(x) \longrightarrow F'_+(x) - F'_\varepsilon(x), \quad G'_{n,\varepsilon}(x) \longrightarrow F''_+(x) - F''_\varepsilon(x) \quad (n \rightarrow \infty)$$

for all $x \neq 0$ and $\varepsilon \in (0, 1)$. Thus, in a same way as the proof of Proposition 3.1, we can obtain

$$\begin{aligned}
& E \left| \int_0^t [F'_+(B_s^H - a) - F'_\varepsilon(B_s^H - a)] \delta B_s^H \right|^2 \\
&= \int_0^t \int_0^t \Lambda_3(s, r, a) \phi(s, r) ds dr + \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \Lambda_4(s, r, a) \phi(s, \eta) \phi(r, \xi) \\
&= \int_0^t \int_0^t \Lambda_3(s, r, a) \phi(s, r) ds dr + \int_0^t \int_0^t (sr)^{2H-1} dr ds \Lambda_4(s, r, a) \phi(s, \eta) \phi(r, \xi) \\
&\leq C_H (t^H + t^{2-(2+\beta)H}) \varepsilon^{\beta \wedge H} (1 + \log^2 \varepsilon) \longrightarrow 0
\end{aligned}$$

with $0 < \beta < \frac{1-H}{H}$ by Lemma 4.1. It follows from the Itô formula (4.7) that

$$\begin{aligned}
(4.12) \quad C_t^{+,H}(a) &= 2 \lim_{\varepsilon \downarrow 0} \left\{ F_\varepsilon(B_t^H - a) - F_\varepsilon(-a) - \int_0^t F'_\varepsilon(B_s^H - a) \delta B_s^H \right\} \\
&= \lim_{\varepsilon \downarrow 0} J_t^H(\varepsilon, a)
\end{aligned}$$

in $L^2(\Omega)$, where

$$J_t^H(\varepsilon, a) = \frac{\log \varepsilon}{\varepsilon} \int_0^t 1_{\{0 \leq B_s^H - a < \varepsilon\}} 2H s^{2H-1} ds + \int_0^t 1_{\{B_s^H - a \geq \varepsilon\}} \frac{2H s^{2H-1}}{B_s^H - a} ds.$$

Step III. To end the proof, we decompose $J_t^H(\varepsilon, a)$ as

$$J_t^H(\varepsilon, a) = I_t(\varepsilon, a) + \left\{ \int_0^t 1_{\{B_s^H - a \geq \varepsilon\}} \frac{2H s^{2H-1}}{B_s^H - a} ds + (\log \varepsilon) \mathcal{L}^H(a, t) \right\},$$

where

$$I_t(\varepsilon, a) := \frac{\log \varepsilon}{\varepsilon} \int_0^t 1_{\{0 \leq B_s^H - a < \varepsilon\}} 2H s^{2H-1} ds - (\log \varepsilon) \mathcal{L}^H(a, t).$$

According to Lemma 4.2 we get

$$\begin{aligned}
E |I_t(\varepsilon, a)|^2 &:= (\log \varepsilon)^2 E \left| \frac{1}{\varepsilon} \int_0^t 1_{\{0 \leq B_s^H - a < \varepsilon\}} 2H s^{2H-1} ds - \mathcal{L}^H(a, t) \right|^2 \\
&= (\log \varepsilon)^2 E \left| \frac{1}{\varepsilon} \int_0^\varepsilon \mathcal{L}^H(x + a, t) dx - \mathcal{L}^H(a, t) \right|^2 \\
&\leq (\log \varepsilon)^2 \frac{1}{\varepsilon} \int_0^\varepsilon E |\mathcal{L}^H(x + a, t) - \mathcal{L}^H(a, t)|^2 dx \\
&\leq C_{H,t,\alpha} \varepsilon^\alpha (\log \varepsilon)^2 \longrightarrow 0 \quad (\varepsilon \rightarrow 0)
\end{aligned}$$

for all $0 < \alpha < \frac{1-H}{H}$ and $t \geq 0$, which shows that

$$C_t^{+,H}(a) = \lim_{\varepsilon \downarrow 0} \left\{ \int_0^t 1_{\{B_s^H - a \geq \varepsilon\}} \frac{2H s^{2H-1}}{B_s^H - a} ds + (\log \varepsilon) \mathcal{L}^H(a, t) \right\} \quad \text{in } L^2(\Omega)$$

for all $t \geq 0$, and the theorem follows. \square

Theorem 4.2. *The convergence*

$$(4.13) \quad C_t^H(a) = \lim_{\varepsilon \downarrow 0} \int_0^t 1_{\{B_s^H - a \geq \varepsilon\}} \frac{2H s^{2H-1}}{B_s^H - a} ds \equiv \text{v.p.} \int_0^t \frac{2H s^{2H-1}}{B_s^H - a} ds$$

holds in $L^2(\Omega)$.

Proof. In the same way as the proof of (4.6), we can show that the convergence

$$(4.14) \quad \mathcal{C}_t^{-,H}(a) = \lim_{\varepsilon \downarrow 0} \left\{ -(\log \varepsilon) \mathcal{L}^H(a, t) + \int_0^t 1_{\{B_s^H - a \leq -\varepsilon\}} \frac{2Hs^{2H-1}}{B_s^H - a} ds \right\}$$

holds in $L^2(\Omega)$. Thus, (4.13) follows from $F = F_+ + F_-$, where $F(x) = x \log |x| - x$. \square

According to the occupation formula we get

$$(4.15) \quad \begin{aligned} \mathcal{C}_t^H(a) &= \lim_{\varepsilon \downarrow 0} \int_0^t 1_{\{|B_s^H - a| \geq \varepsilon\}} \frac{2Hs^{2H-1}}{B_s^H - a} ds \quad \text{in } L^2(\Omega) \\ &= \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} 1_{\{|x-a| \geq \varepsilon\}} \frac{\mathcal{L}^H(x, t)}{x - a} dx \quad \text{in } L^2(\Omega) \\ &= \text{v.p.} \int_{\mathbb{R}} \frac{\mathcal{L}^H(x, t)}{x - a} dx = \pi \left(\mathcal{H} \mathcal{L}^H(\cdot, t) \right) (a) \end{aligned}$$

for all $t \geq 0$ and $a \in \mathbb{R}$. As two natural results we get the fractional version of Yamada's formula

$$\begin{aligned} (B_t^H - a) \log |B_t^H - a| - (B_t^H - a) \\ = -a \log |a| + a + \int_0^t \log |B_s^H - a| \delta B_s^H + \frac{1}{2} \text{v.p.} \int_{\mathbb{R}} \frac{\mathcal{L}^H(x, t)}{x - a} dx \end{aligned}$$

for all $t \geq 0$ and $a \in \mathbb{R}$, and

$$\begin{aligned} \mathcal{C}_t^H(b) - \mathcal{C}_s^H(a) \\ = \int_0^\infty \left[\mathcal{L}^H(b+x, t) - \mathcal{L}^H(b-x, t) - \mathcal{L}^H(a+x, s) + \mathcal{L}^H(a-x, s) \right] \frac{dx}{x} \end{aligned}$$

for all $a, b \in \mathbb{R}$ and $s, t \geq 0$. Recall that the local time $\mathcal{L}^H(x, t)$ admits a compact support and it is Hölder continuous of order $\gamma \in (0, 1 - H)$ in time, and of order $\kappa \in (0, \frac{1-H}{2H})$ in the space variable (see Geman-Horowitz [16]). We see that the process $(a, t) \mapsto \mathcal{C}_t^H(a)$ admits Hölder continuous paths. In particular, we have

Proposition 4.1. *Let $\frac{1}{2} < H < 1$. For all $t' > t \geq 0$, we have*

$$E \left[|\mathcal{C}_{t'}^H - \mathcal{C}_t^H|^2 \right] \leq C(t' - t)^{2H_0},$$

where

$$H_0 = \begin{cases} H, & \text{if } \frac{1}{2} < H \leq \frac{2}{3}, \\ 1 - \frac{1}{2}H, & \text{if } \frac{2}{3} < H < 1. \end{cases}$$

Proof. Given $\varepsilon > 0$ and denote

$$\mathcal{C}_t^{H, \varepsilon} = \int_0^t 1_{\{|B_s^H| > \varepsilon\}} \frac{ds^{2H}}{B_s^H}$$

for $t \geq 0$. We have

$$\begin{aligned}
E \left[1_{\{|B_s^H| > \varepsilon\}} 1_{\{|B_r^H| > \varepsilon\}} \frac{1}{B_s^H B_r^H} \right] &= \int_{\mathbb{R}^2} 1_{\{|x| > \varepsilon\}} 1_{\{|y| > \varepsilon\}} \frac{1}{xy} \varphi_{s,r}(x, y) dx dy \\
&= \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \frac{1}{xy} \varphi_{s,r}(x, y) dx dy + \int_{-\infty}^{-\varepsilon} \int_{\varepsilon}^{\infty} \frac{1}{xy} \varphi_{s,r}(x, y) dx dy \\
&\quad + \int_{\varepsilon}^{\infty} \int_{-\infty}^{-\varepsilon} \frac{1}{xy} \varphi_{s,r}(x, y) dx dy + \int_{-\infty}^{-\varepsilon} \int_{-\infty}^{-\varepsilon} \frac{1}{xy} \varphi_{s,r}(x, y) dx dy \\
&= \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \frac{1}{xy} [\varphi_{s,r}(x, y) - \varphi_{s,r}(-x, y) - \varphi_{s,r}(x, -y) + \varphi_{s,r}(-x, -y)] dx dy \\
&= 2 \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \frac{1}{xy} [\varphi_{s,r}(x, y) - \varphi_{s,r}(-x, y)] dx dy \\
&= 2 \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \frac{1}{xy} \left(1 - e^{-\frac{1}{\rho_{s,r}^2} \mu_{s,r} xy} \right) \varphi_{s,r}(x, y) dx dy \\
&= 2 \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} \left(\int_0^{\frac{\mu_{s,r}}{\rho_{s,r}^2}} e^{-xy\xi} d\xi \right) \varphi_{s,r}(x, y) dx dy \\
&= 2 \int_0^{\frac{\mu_{s,r}}{\rho_{s,r}^2}} d\xi \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} e^{-xy\xi} \varphi_{s,r}(x, y) dx dy
\end{aligned}$$

for all $s, t \geq 0$. An elementary calculus can show that

$$\begin{aligned}
&\int_0^{\infty} \int_0^{\infty} e^{-xy\xi} \varphi_{s,r}(x, y) dx dy \\
&= \frac{1}{2\pi\rho_{s,r}} \int_0^{\infty} e^{-\frac{1}{r^{2H}}(1+2\mu_{s,r}\xi-\rho_{s,r}^2\xi^2)y^2} dy \int_0^{\infty} e^{-\frac{r^{2H}}{2\rho_{s,r}^2}\left(x-\frac{1}{r^{2H}}(\mu_{s,r}-\rho_{s,r}^2\xi)y\right)^2} dx \\
&= \frac{1}{4\sqrt{1+2\mu_{s,r}\xi-\rho_{s,r}^2\xi^2}}
\end{aligned}$$

for all $\xi > 0$, which implies that

$$\begin{aligned}
E \left[1_{\{|B_s^H| > \varepsilon\}} 1_{\{|B_r^H| > \varepsilon\}} \frac{1}{B_s^H B_r^H} \right] &= \int_{\mathbb{R}^2} 1_{\{|x| > \varepsilon\}} 1_{\{|y| > \varepsilon\}} \frac{1}{xy} \varphi_{s,r}(x, y) dx dy \\
&\leq \int_0^{\frac{\mu_{s,r}}{\rho_{s,r}^2}} \frac{1}{4\sqrt{1+2\mu_{s,r}\xi-\rho_{s,r}^2\xi^2}} d\xi = \frac{1}{4\rho_{s,r}} \arcsin \frac{\mu_{s,r}}{\sqrt{\rho_{s,r}^2 + \mu_{s,r}^2}} \\
&= \frac{1}{\rho_{s,r}} \arcsin \frac{\mu_{s,r}}{(sr)^H} \leq \frac{1}{\rho_{s,r}}.
\end{aligned}$$

It follows that

$$E \left[|\mathcal{C}_{t'}^{H,\varepsilon} - \mathcal{C}_t^{H,\varepsilon}|^2 \right] \leq \int_t^{t'} \int_t^{t'} \frac{1}{\rho_{s,r}} ds^{2H} dr^{2H} \leq \begin{cases} C(t' - t)^{2H}, & \text{if } \frac{1}{2} < H \leq \frac{2}{3}, \\ C(t - t')^{2-H}, & \text{if } \frac{2}{3} < H < 1 \end{cases}$$

for all $0 < t < t' < T$ and $\varepsilon > 0$. This shows that

$$E \left[|\mathcal{C}_{t'}^H - \mathcal{C}_t^H|^2 \right] \leq C(t' - t)^{2H_0}$$

and the proposition follows. \square

Remark 4.1. The above continuity results for the process $(x, t) \mapsto \mathcal{C}^H(x, t) := \mathcal{C}_t^H(x)$ are some reminders to us that we may consider the following integrals:

$$\int_0^t u_s d\mathcal{C}_s^H, \quad \int_{\mathbb{R}} f(x) \mathcal{C}^H(dx, t), \quad \int_0^t \int_{\mathbb{R}} f(x, s) \mathcal{C}^H(dx, ds),$$

where u is an adapted process, and $(x, t) \mapsto f(x, t)$ and $x \mapsto f(x)$ Borel functions on $\mathbb{R} \times [0, T]$ and \mathbb{R} , respectively. These will be considered in the other paper.

5. THE OCCUPATION FORMULA ASSOCIATED WITH $\mathcal{C}^H(a)$

From the previous sections we know that the process $(a, t) \mapsto \mathcal{C}_t^H(a)$ is Hölder continuous and in this section our main object is to expound and prove the next theorem which is an analogue of the occupation formula.

Theorem 5.1. *Let $\frac{1}{2} < H < 1$ and let g be a continuous function with compact support. We then have, almost surely,*

$$(5.1) \quad \int_{\mathbb{R}} \mathcal{C}_t^H(x) g(x) dx = 2H\pi \int_0^t (\mathcal{H}g)(B_s^H) s^{2H-1} ds$$

and

$$2H\pi \int_0^t f(B_s^H) s^{2H-1} ds = \int_{\mathbb{R}} \mathcal{C}_t^H(x) (\mathcal{H}^{-1}f)(x) dx$$

for all $t \geq 0$, where the operator \mathcal{H}^{-1} means the inverse transform of Hilbert transform \mathcal{H} .

In order to prove the theorem we need some preliminaries.

Lemma 5.1. *Let $F(x) = x \log|x| - x$ and let g be a continuous function with compact support. Then the integral*

$$\int_0^t (F' * g)(B_s^H) \delta B_s^H$$

exists in $L^2(\Omega)$ for all $t \geq 0$ and the process

$$\mathcal{X}_t^g := (F * g)(B_t^H) - (F * g)(0) - \int_0^t (F' * g)(B_s^H) \delta B_s^H, \quad t \geq 0$$

is well-defined.

Proof. From Lemma 3.1 it follows that

$$\begin{aligned} E \left| \int_0^t (G * g)(B_s^H) \delta B_s^H \right|^2 &= \int_0^t \int_0^t E [(G * g)(B_s^H) (G * g)(B_r^H)] \phi(s, r) ds dr \\ &\quad + \int_0^t ds \int_0^t dr \int_0^s d\xi \int_0^r d\eta \phi(s, \eta) \phi(r, \xi) E [(G' * g)(B_s^H) (G' * g)(B_r^H)] \\ &= \int_0^t \int_0^t \phi(s, r) ds dr \int_{\mathbb{R}^2} g(u) g(v) du dv \int_{\mathbb{R}^2} G(x - u) G(y - v) \varphi_{s,r}(x, y) dx dy \\ &\quad + \int_0^t ds \int_0^t dr \int_0^s d\xi \int_0^r d\eta \phi(s, \eta) \phi(r, \xi) \\ &\quad \cdot \int_{\mathbb{R}^2} g(u) g(v) du dv \int_{\mathbb{R}^2} G'(x - u) G'(y - v) \Psi_{s,r,u,v}(x, y) dx dy \\ &\quad + \int_0^t ds \int_0^t dr \int_0^s d\xi \int_0^r d\eta \phi(s, \eta) \phi(r, \xi) \int_{\mathbb{R}^2} g(u) g(v) \Lambda_7(s, r, u, v) du dv \end{aligned}$$

for all $t > 0, u, v \in \mathbb{R}$, and all $G \in C^\infty(\mathbb{R})$ with compact support, where

$$\begin{aligned} \Lambda_7(s, r, u, v) &= -G(1) \int_{\mathbb{R}} G(x - u) \frac{\partial}{\partial x} \varphi_{s,r}(x, v) dx \\ &\quad - G(1) \int_{\mathbb{R}} G(y - v) \frac{\partial}{\partial y} \varphi_{s,r}(u, y) dy - \varphi_{s,r}(u, v) G(1) G(1). \end{aligned}$$

Decompose F as

$$F(x) = F_+(x) + F_-(x),$$

where F_+ and F_- are given in Section 3. Clearly, we have

$$\begin{aligned}
& \int_{\mathbb{R}^2} |g(u)g(v)| dudv \int_{\mathbb{R}^2} |F'_+(x-u)F'_+(y-v)| \varphi_{s,r}(x,y) dx dy \\
& \leq \int_{\mathbb{R}^2} |g(u)g(v)| dudv \left(\int_u^\infty \log^2(x-u) \varphi_s(x) dx \int_v^\infty \log^2(y-v) \varphi_r(y) dy \right)^{1/2} \\
& \leq \int_{\mathbb{R}^2} |g(u)g(v)| dudv (s^{-H} + s^H + |u|)^{1/2} (r^{-H} + r^H + |v|)^{1/2} \\
& \leq C_H (r^{-H} + s^H + 1) \left(\int_{\mathbb{R}} |g(u)| (\sqrt{|u|} + 1) du \right)^2,
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\mathbb{R}^2} |g(u)g(v)| dudv \int_u^\infty dx \int_v^\infty \frac{|\Psi_{s,r,u,v}(x,y)| dy}{(x-u)(y-v)} \\
& \leq C_{H,T,\beta} \frac{s^{\beta H/2}}{r^{(1+\beta)H} (s-r)^{(1+\beta)H}} \int_{\mathbb{R}^2} |g(u)g(v)| dudv
\end{aligned}$$

by Lemma 3.3. Thus, similar to the proof of Proposition 3.1 by approximating the function $F'_+(x)$ by smooth functions with compact support, we can show that the integral $\int_0^t (F'_+ * g)(B_s^H) \delta B_s^H$ exists in $L^2(\Omega)$ for all $t \geq 0$ and

$$\begin{aligned}
E \left| \int_0^t (F'_+ * g)(B_s^H) \delta B_s^H \right|^2 &= \int_0^t \int_0^t ds dr \phi(s,r) \int_{\mathbb{R}^2} dudv g(u)g(v) \\
&\quad \cdot \int_{\mathbb{R}^2} F'_+(x-u) F'_+(y-v) \varphi_{s,r}(x,y) dx dy \\
&\quad + \int_0^t ds \int_0^s d\xi \int_0^t dr \int_0^r d\eta \phi(s,\eta) \phi(r,\xi) \\
&\quad \cdot \int_{\mathbb{R}^2} g(u)g(v) dudv \int_u^\infty dx \int_v^\infty \frac{\Psi_{s,r,u,v}(x,y) dy}{(x-u)(y-v)}.
\end{aligned}$$

Similarly, we can also show that the integral $\int_0^t (F'_- * g)(B_s^H) \delta B_s^H$ exists in $L^2(\Omega)$ for all $t \geq 0$, and the lemma follows since $F = F_+ + F_-$. \square

Lemma 5.2. *Let $F(x) = x \log |x| - x$ and let g be a continuous function with compact support. Then*

$$(5.2) \quad \int_0^t \left(\int_{\mathbb{R}} F'(B_s^H - x) g(x) dx \right) \delta B_s^H = \int_{\mathbb{R}} \left(\int_0^t F'(B_s^H - x) \delta B_s^H \right) g(x) dx$$

for all $0 \leq t \leq T$.

By using the divergence operator δ^H we can rewrite (5.2) as

$$\delta^H(F' * g(B^H)) = \int_{\mathbb{R}} g(a) da \int_0^t F'(B^H - a) \delta B_s^H.$$

Proof of Lemma 5.2. Clearly, we have

$$F' * g = (F * g)', \quad (F * g)'' = \text{v.p.} \frac{1}{x} * g.$$

Moreover, the functional

$$x \mapsto \int_0^t F'(B_s^H - x) \delta B_s^H$$

is Borel measurable for every $t \geq 0$ and the right-hand side in (5.2) exists also in $L^2(\Omega)$ by Proposition 3.1.

Denote by X the process concerning the right hand in (5.2) and let

$$u_t = \int_{\mathbb{R}} F'(B_t^H - x)g(x)dx$$

for $t \geq 0$. Then, the process X and u are measurable. Thus, it is enough to show that the following duality relationship holds:

$$(5.3) \quad E[UX_T] = E[\langle D^H U, u \rangle_{\mathcal{H}}]$$

for all $U \in \mathbb{D}^{1,2}$ by Lemma 5.1. This is clear. In fact, noting that

$$\begin{aligned} \int_{\mathbb{R}} \left(\int_0^T \int_0^T (D_s^H U) F'(B_r^H - x) \phi(s, r) ds dr \right) g(x) dx \\ = \int_0^T \int_0^T (D_s^H U) \left(\int_{\mathbb{R}} F'(B_r^H - x) g(x) dx \right) \phi(s, r) ds dr, \quad \text{a.s.} \end{aligned}$$

for all $U \in \mathbb{D}^{1,2}$, we have

$$\begin{aligned} E[UX_T] &= E \left[U \int_{\mathbb{R}} \left(\int_0^T F'(B_s^H - x) \delta B_s^H \right) g(x) dx \right] \\ &= \int_{\mathbb{R}} E \left[U \int_0^T F'(B_s^H - x) \delta B_s^H \right] g(x) dx \\ &= \int_{\mathbb{R}} (E \langle D^H U, F'(B^H - x) \rangle_{\mathcal{H}}) g(x) dx \\ &= E \int_{\mathbb{R}} \left(\int_0^T \int_0^T (D_s^H U) F'(B_r^H - x) \phi(s, r) ds dr \right) g(x) dx \\ &= E \int_0^T \int_0^T (D_s^H U) \left(\int_{\mathbb{R}} F'(B_r^H - x) g(x) dx \right) \phi(s, r) ds dr = E[\langle D^H U, u \rangle_{\mathcal{H}}] \end{aligned}$$

for all $U \in \mathbb{D}^{1,2}$, and the lemma follows. \square

Proof of Theorem 5.1. Let $F(x) = x \log |x| - x$. Then second derivative $(F * g)'' = F'' * g$ exists in the sense of Schwartz's distribution, and similar to Theorem 4.1 we have

$$\mathcal{X}_t^g = H \int_0^t \text{v.p.} \frac{1}{x} * g(B_s^H) s^{2H-1} ds$$

for all $t \geq 0$, where \mathcal{C}^g is defined in Lemma 5.1. It follows from Lemma 5.2 that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}} \mathcal{C}_t^H(x) g(x) dx &= \int_{\mathbb{R}} \left(F(B_t^H - x) - F(-x) - \int_0^t F'(B_s^H - x) \delta B_s^H \right) g(x) dx \\ &= F * g(B_t^H) - F * g(0) - \int_0^t F' * g(B_s^H) \delta B_s^H \\ &= F * g(B_t^H) - F * g(0) - \int_0^t (F * g)'(B_s^H) \delta B_s^H \\ &= H \int_0^t \text{v.p.} \frac{1}{x} * g(B_s^H) s^{2H-1} ds \\ &= H \pi \int_0^t \mathcal{H} g(B_s^H) s^{2H-1} ds \end{aligned}$$

for all $t \geq 0$. This completes the proof. \square

Corollary 5.1. Let $\frac{1}{2} < H < 1$ and let $g, g_n \in L^2(\mathbb{R})$ be continuous with compact supports. If $g_n \rightarrow g$ in $L^2(\mathbb{R})$, as n tends to infinity, we then have

$$(5.4) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \mathcal{C}_t^H(x) g_n(x) dx = \int_{\mathbb{R}} \mathcal{C}_t^H(x) g(x) dx$$

for all $t \geq 0$, in the $L^2(\Omega)$.

Proof. The convergence follows from the identity

$$\int_{\mathbb{R}} (g_n(x) - g(x))^2 dx = \int_{\mathbb{R}} (\mathcal{H} g_n(x) - \mathcal{H} g(x))^2 dx.$$

and Theorem 5.1. \square

6. THE CASE $0 < H < \frac{1}{2}$

In the final section we consider the process \mathcal{C}^H with $0 < H < \frac{1}{2}$. Recall that for $0 < H < \frac{1}{2}$, Yan *et al* [33] obtained the *generalized quadratic covariation* of $f(B^H)$ and B^H defined by

$$[f(B^H), B^H]_t^{(H)} := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2H}} \int_0^t \{f(B_{s+\varepsilon}^H) - f(B_s^H)\} (B_{s+\varepsilon}^H - B_s^H) ds^{2H}$$

in probability, where f is a Borel function. In Yan *et al* [33] one constructed the Banach space $\mathcal{H} = L^2(\mathbb{R}, \mu(dx))$ with

$$\mu(dx) = \left(\int_0^T e^{-\frac{x^2}{2s^{2H}}} \frac{2H ds}{\sqrt{2\pi s^{1-H}}} \right) dx$$

and

$$\|f\|_{\mathcal{H}}^2 = \int_0^T \int_{\mathbb{R}} |f(x)|^2 e^{-\frac{x^2}{2s^{2H}}} \frac{2H dx ds}{\sqrt{2\pi s^{1-H}}} = E \left(\int_0^T |f(B_s^H)|^2 ds^{2H} \right),$$

such that the generalized quadratic covariation $[f(B^H), B^H]^{(H)}$ exists in $L^2(\Omega)$ and

$$(6.1) \quad E \left| [f(B^H), B^H]_t^{(H)} \right|^2 \leq C \|f\|_{\mathcal{H}}^2,$$

provided $f \in \mathcal{H}$. Moreover, the Bouleau-Yor identity takes the form

$$[f(B^H), B^H]_t^{(H)} = - \int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t)$$

for all $f \in \mathcal{H}$. By using the generalized quadratic covariation Yan *et al* [33] obtained the next Itô formula:

$$(6.2) \quad F(B_t^H) = F(0) + \int_0^t f(B_s^H) \delta B_s^H + \frac{1}{2} [f(B^H), B^H]_t^{(H)}$$

for all $0 < H < \frac{1}{2}$, where F is an absolutely continuous function such that $F' = f \in \mathcal{H}$ is left (right) continuous. It is important to note that the method used in Yan *et al* [33] is inefficacy for $\frac{1}{2} < H < 1$ in general and the similar results for $\frac{1}{2} < H < 1$ is unknown so far.

Corollary 6.1. *Let $0 < H < \frac{1}{2}$ and let $F(x) = x \log |x| - x$. Then $F' \in \mathcal{H}$ and the Itô type formula*

$$(6.3) \quad F(B_t^H - a) = F(-a) + \int_0^t \log |B_s^H - a| \delta B_s^H + \frac{1}{2} [\log(B^H - a), B^H]_t^{(H)}$$

holds and

$$(6.4) \quad \mathcal{C}_t^H(a) = [\log |B^H - a|, B^H]_t^{(H)}$$

for all $t \geq 0$ and $a \in \mathbb{R}$.

Proof. Let F_+ and F_- be defined in Section 3. Then $F'_+ \in \mathcal{H}$ is left continuous, and

$$(6.5) \quad F_+(B_t^H - a) = F_+(-a) + \int_0^t F'_+(B_s^H - a) \delta B_s^H + \frac{1}{2} [F'_+(B^H - a), B^H]_t^{(H)}$$

by Itô's formula (6.2). Similarly, we have

$$(6.6) \quad F_-(B_t^H - a) = F_-(-a) + \int_0^t F'_-(B_s^H - a) \delta B_s^H + \frac{1}{2} [F'_-(B^H - a), B^H]_t^{(H)}$$

since $F'_- \in \mathcal{H}$ is right continuous. Thus, the corollary follows from $F = F'_+ + F'_-$. \square

Denote

$$\begin{aligned}\Theta_\varepsilon(t, a) &:= \mathcal{L}^H(a - \varepsilon, t)F'(-\varepsilon) - \mathcal{L}^H(a + \varepsilon, t)F'(\varepsilon) \\ &= [\mathcal{L}^H(a - \varepsilon, t) - \mathcal{L}^H(a + \varepsilon, t)] \log \varepsilon.\end{aligned}$$

By integration by parts we have

$$\begin{aligned}\mathcal{C}_t^H(a) &= [\log |B^H - a|, B^H]_t^{(H)} = - \int_{\mathbb{R}} \log |x - a| \mathcal{L}^H(dx, t) \\ &= - \lim_{\varepsilon \downarrow 0} \left(\int_{a+\varepsilon}^{\infty} \log |x - a| \mathcal{L}^H(dx, t) + \int_{-\infty}^{a-\varepsilon} \log |x - a| \mathcal{L}^H(dx, t) \right) \\ &= \lim_{\varepsilon \downarrow 0} \left(\int_{a+\varepsilon}^{\infty} \frac{\mathcal{L}^H(x, t)}{x - a} dx + \int_{-\infty}^{a-\varepsilon} \frac{\mathcal{L}^H(x, t)}{x - a} dx \right) + \lim_{\varepsilon \downarrow 0} \Theta_\varepsilon(t, a) \\ &= \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} 1_{\{|x-a| \geq \varepsilon\}} \frac{\mathcal{L}^H(x, t)}{x - a} dx \\ &= \text{v.p.} \int_{\mathbb{R}} \frac{\mathcal{L}^H(x, t)}{x - a} dx = \pi \mathcal{H} \mathcal{L}^H(\cdot, t)(x)\end{aligned}$$

almost surely and in $L^2(\Omega)$, for all $t \geq 0$ and $a \in \mathbb{R}$ since $x \mapsto \mathcal{L}^H(x, \cdot)$ is Hölder continuous and has the compact support.

Corollary 6.2. *Let F be given by Corollary 6.1 and let g be a continuous function with compact support. Then the integral $F' * g \in \mathcal{H}$, and for all $0 < H < \frac{1}{2}$, the process*

$$2 \left((F * g)(B_t^H) - (F * g)(0) - \int_0^t (F' * g)(B_s^H) \delta B_s^H \right)$$

is well-defined in $L^2(\Omega)$ and is equal to

$$[(F' * g)(B^H), B^H]_t^{(H)}$$

for all $t \in [0, T]$.

Thus, similar to proof of Theorem 5.1 we can obtain the following occupation formula.

Theorem 6.1. *Let $0 < H < \frac{1}{2}$ and let g be a continuous function with compact support. Then we have, almost surely,*

$$(6.7) \quad \int_{\mathbb{R}} \mathcal{C}_t^H(x) g(x) dx = 2H\pi \int_0^t (\mathcal{H}g)(B_s^H) s^{2H-1} ds$$

and

$$2H\pi \int_0^t g(B_s^H) s^{2H-1} ds = \int_{\mathbb{R}} \mathcal{C}_t^H(x) (\mathcal{H}^{-1}g)(x) dx$$

for all $t \in [0, T]$.

Proof. Let $F(x) = x \log |x| - x$. By Corollary 6.1 and (6.1) we have

$$E \left| \int_{\mathbb{R}} \left(\int_0^t F'(B_s^H - x) \delta B_s^H \right) g(x) dx \right|^2 < \infty,$$

since g admits a compact support. Thus, similar to Lemma 5.2 we can show that the Fubini theorem

$$(6.8) \quad \int_0^t \left(\int_{\mathbb{R}} F'(B_s^H - x) g(x) dx \right) \delta B_s^H = \int_{\mathbb{R}} \left(\int_0^t F'(B_s^H - x) \delta B_s^H \right) g(x) dx$$

holds for all $0 \leq t \leq T$. It follows from Corollary 6.1 and Corollary 6.2 that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}} \mathcal{C}_t^H(x) g(x) dx &= \int_{\mathbb{R}} \left(F(B_s^H - x) - F(-x) - \int_0^t F'(B_s^H - x) \delta B_s^H \right) g(x) dx \\ &= F * g(B_t^H) - F * g(0) - \int_0^t F' * g(B_s^H) \delta B_s^H \\ &= \frac{1}{2} [(F' * g)(B^H), B^H]_t^{(H)} \\ &= -\frac{1}{2} \int_{\mathbb{R}} (F' * g)(x) \mathcal{L}^H(dx, t) \end{aligned}$$

On the other hand, by the Hölder continuity of $(x, t) \mapsto \mathcal{L}^H(x, t)$ and Lebesgue's dominated convergence theorem we have

$$\begin{aligned} &\int_{\mathbb{R}} g(a) da \int_{\mathbb{R}} F'(x - a) \mathcal{L}^H(dx, t) \\ &= \int_{\mathbb{R}} g(a) \lim_{\varepsilon \downarrow 0} \left(\int_{a+\varepsilon}^{\infty} F'(x - a) \mathcal{L}^H(dx, t) + \int_{-\infty}^{a-\varepsilon} F'(x - a) \mathcal{L}^H(dx, t) \right) da \\ &= \int_{\mathbb{R}} g(a) \lim_{\varepsilon \downarrow 0} \left(\Theta_{\varepsilon}(t, a) - \int_{\mathbb{R}} 1_{\{|x-a|>\varepsilon\}} F''(x - a) \mathcal{L}^H(x, t) dx \right) da \\ &= -2H \int_{\mathbb{R}} g(a) \lim_{\varepsilon \downarrow 0} \left(\int_0^t 1_{\{|B_s^H - a|>\varepsilon\}} F''(B_s^H - a) s^{2H-1} ds \right) da \\ &= -2H \lim_{\varepsilon \downarrow 0} \int_0^t s^{2H-1} ds \int_{\mathbb{R}} 1_{\{|B_s^H - a|>\varepsilon\}} \frac{g(a)}{B_s^H - a} da \\ &= -2H\pi \int_0^t \mathcal{H} g(B_s^H) s^{2H-1} ds \end{aligned}$$

almost surely and in $L^2(\Omega)$, for all $t \in [0, T]$. This shows that

$$\frac{1}{2} \int_{\mathbb{R}} \mathcal{C}_t^H(x) g(x) dx = H\pi \int_0^t \mathcal{H} g(B_s^H) s^{2H-1} ds$$

and the theorem follows. \square

Remark 6.1. When $0 < H < \frac{1}{2}$, from the discussion in this section, we have found that for all non-locally integrable Borel functions $f \in \mathcal{H}$, the identities

$$\mathcal{K}_t^H(f, a) := \lim_{\varepsilon \downarrow 0} \int_0^t 1_{\{|B_s^H - a|>\varepsilon\}} f(B_s^H) ds^{2H} = [f(B^H), B^H]_t^{(H)}$$

in $L^2(\Omega)$ (almost surely) and

$$\int_{\mathbb{R}} \mathcal{K}_t^H(f, x) g(x) dx = 2H \int_0^t \text{v.p.}(f' * g)(B_s^H) s^{2H-1} ds$$

hold for all continuous functions g with compact supports, provided

$$\mathcal{L}^H(a - \varepsilon, t) f(-\varepsilon) - \mathcal{L}^H(a + \varepsilon, t) f(\varepsilon) \longrightarrow 0,$$

in $L^2(\Omega)$ (almost surely), as ε tends to zero.

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